

Freezing and Decorated Poisson Point Processes

Eliran Subag, Ofer Zeitouni

Weizmann Institute of Science, Rehovot, Israel. E-mail: eliran.subag@weizmann.ac.il; ofer.zeitouni@weizmann.ac.il

Received: 1 May 2014 / Accepted: 15 September 2014
Published online: 26 February 2015 – © Springer-Verlag Berlin Heidelberg 2015

Abstract: The limiting extremal processes of the branching Brownian motion (BBM), the two-speed BBM, and the branching random walk are known to be randomly shifted decorated Poisson point processes (SDPPP). In the proofs of those results, the Laplace functional of the limiting extremal process is shown to satisfy $L[\theta_y f] = g(y - \tau_f)$ for any nonzero, nonnegative, compactly supported, continuous function $f$, where $\theta_y$ is the shift operator, $\tau_f$ is a real number that depends on $f$, and $g$ is a real function that is independent of $f$. We show that, under some assumptions, this property characterizes the structure of SDPPP. Moreover, when it holds, we show that $g$ has to be a convolution of the Gumbel distribution with some measure.

The above property of the Laplace functional is closely related to a ‘freezing phenomenon’ that is expected to occur in a wide class of log-correlated fields, and which has played an important role in the analysis of various models. Our results shed light on this intriguing phenomenon and provide a natural tool for proving an SDPPP structure in these and other models.

1. Introduction

The branching Brownian motion (BBM) is a continuous-time branching process described as follows. At time $t = 0$ a single particle starts a standard Brownian motion at distance $x$ from the origin, continuing for a randomly distributed exponential time $T$ independent of $x$. At this moment, the particle splits into two particles. Each, in turn, performs a Brownian motion starting from $x(T)$ and is subject to the same splitting rule. Thus, at time $t$ there is a random number of particles $N(t)$ and we denote their positions by $X_1(t), \ldots, X_{N(t)}(t)$.

The BBM has been extensively studied over the last decades. The seminal works of McKean [45], Bramson [12,13], and Lalley and Sellke [40] were mainly concerned

Research partially supported by a grant of the Israel Science Foundation.
with the maximum (or rightmost particle) \( M_t = \max_{i \leq N(t)} X_i(t) \) and its relations to the Fisher—Kolmogorov–Petrovsky–Piscounov (F-KPP) equation [39]. In particular, it was shown in [40] that with appropriate recentering term \( m_t \),

\[
\lim_{t \to \infty} \mathbb{P} \{ M_t - m_t \leq x \} = \mathbb{E} \exp \left\{ -e^{-\sqrt{2} (x - \log(CZ)/\sqrt{2})} \right\},
\]

\[1.1\]

where \( Z \) is the limit of the so-called derivative martingale and \( C \) is a constant.

Recently, BBM became the object of renewed interest with the main focus being the behavior of extreme values of the process [1,3–6,15]. Perhaps the most important result in this direction is a remarkable description of the limiting extremal process, i.e., the limit in distribution \( \xi = \lim_{t \to \infty} \xi_t \triangleq \lim_{t \to \infty} \sum_{i \leq N(t)} \delta_{X_i(t) - m_t} \),

\[1.2\]

given independently by Arguin et al. [6] and Aïdékon et al. [1].

In the sequel, we denote by \( d = \) equality in distribution. For a point process \( D = \sum_{i \geq 1} \delta_d \), we denote by \( \theta_x D \) the shift of \( D \) by \( x \), i.e. \( \theta_x D = \sum_{i \geq 1} \delta_{d_i + x} \).

The following notions describe the structure of limits alluded to above.

**Definition 1.** (1) A point process \( \psi \) is a decorated Poisson point process (DPPP) of intensity \( \nu \) and decoration \( D \) (denoted \( \psi \sim \text{DPPP}(\nu, D) \)), if \( \psi \overset{d}= \sum_{i \geq 1} \theta_{\xi_i} D_i \) where \( \xi = \sum_{i \geq 1} \delta_{\xi_i} \) is a Poisson process with intensity \( \nu \), \( D_i, \ i \geq 1 \), are copies of \( D \), independent of each other and of \( \xi \).

(2) A point process \( \phi \) is a randomly shifted decorated Poisson point process (SDPPP) of intensity \( \nu \), decoration \( D \) and shift \( S \) (denoted \( \phi \sim \text{SDPPP}(\nu, D, S) \)) if for \( \psi \sim \text{DPPP}(\nu, D) \) and some independent (of \( \psi \)) random variable \( S \) it holds that \( \phi \overset{d}= \theta_S \psi \).

In this notation [1,6] showed that, with some point process \( D \), the limiting extremal process \( \xi \) in (1.2) satisfies

\[
\xi \sim \text{SDPPP} \left( e^{-\sqrt{2}x} dx, D, \log (\sqrt{2}CZ) / \sqrt{2} \right) ,
\]

where \( C, Z \) are as in (1.1).

Our work centers around the relations between two properties of the limiting extremal process, one being the specific structure we have just described. The other is related to an intriguing ‘freezing’ phenomenon observed first by Derrida and Spohn in [25], exploiting Bramson’s results on the F-KPP equation [13]. They define the function \( G_{t,\beta}(y) \overset{\Delta}{=} \mathbb{E} \exp \left\{ -e^{-\beta y} \sum_{i=1}^{N(t)} e^{\beta X_i(t)} \right\} \) and conclude it exhibits the shape of a ‘traveling wave’ as \( t \to \infty \). That is, for some \( m_{t,\beta} \) increasing in \( t \),

\[
\lim_{t \to \infty} G_{t,\beta} \left( y + m_{t,\beta} \right) = g_{\beta} (y) .
\]

Moreover, they show that the profile of the wave and the velocity ‘freeze’ at a certain transition temperature \( \beta = \beta_c \) (\( \beta \) is the inverse temperature):

\[
\text{for any } \beta > \beta_c : \ g_{\beta} (x) = g_{\beta_c} (x) \text{ and } m_{t,\beta} + c_{\beta} = m_t ,
\]

\[1.4\]

with some constants \( c_{\beta} \) depending on \( \beta \).