Linear Response for Intermittent Maps

Viviane Baladi\textsuperscript{1,2}, Mike Todd\textsuperscript{3}

\textsuperscript{1} D.M.A., UMR 8553, École Normale Supérieure, 75005 Paris, France
\textsuperscript{2} Current address: CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche (IMJ-PRG), Analyse Algébrique, Sorbonne Universités, UPMC Univ Paris 06, 4, Place Jussieu, 75005 Paris, France.
E-mail: viviane.baladi@imj-prg.fr
\textsuperscript{3} Mathematical Institute, University of St Andrews, North Haugh, St Andrews, KY16 9SS, Scotland.
E-mail: m.todd@st-andrews.ac.uk

Received: 5 September 2015 / Accepted: 30 November 2015
Published online: 22 February 2016 – © Springer-Verlag Berlin Heidelberg 2016

Abstract: We consider the one parameter family $\alpha \mapsto T_\alpha$ ($\alpha \in [0, 1)$) of Pomeau-Manneville type interval maps $T_\alpha(x) = x(1+2^\alpha x^\alpha)$ for $x \in [0, 1/2)$ and $T_\alpha(x) = 2x - 1$ for $x \in [1/2, 1]$, with the associated absolutely continuous invariant probability measure $\mu_\alpha$. For $\alpha \in (0, 1)$, Sarig and Gouëzel proved that the system mixes only polynomially with rate $n^{1-1/\alpha}$ (in particular, there is no spectral gap). We show that for any $\psi \in L^q$, the map $\alpha \mapsto \int_0^1 \psi \, d\mu_\alpha$ is differentiable on $[0, 1-1/q)$, and we give a (linear response) formula for the value of the derivative. This is the first time that a linear response formula for the SRB measure is obtained in the setting of slowly mixing dynamics. Our argument shows how cone techniques can be used in this context. For $\alpha \geq 1/2$ we need the $n^{1-1/\alpha}$ decorrelation obtained by Gouëzel under additional conditions.

1. Introduction

Given a family of dynamical systems $T_\alpha$ on a Riemann manifold, depending smoothly on a real parameter $\alpha$, and admitting (at least for some large subset of parameters) an ergodic physical (e.g., absolutely continuous, or SRB) invariant measure $\mu_\alpha$, it is natural to ask how smooth is the dependence of $\mu_\alpha$ on the parameter $\alpha$. In particular, one would like to know whether $\alpha \mapsto \mu_\alpha$ is differentiable and, if possible, compute a formula for the derivative, depending on $\mu_\alpha$, $T_\alpha$, and $v_\alpha = \partial_\alpha T_\alpha$.

This theme of linear response was explored in a few pioneering papers [\textit{Ru1}, \textit{KKPW}, \textit{Ru}] in the setting of smooth hyperbolic dynamics (Anosov or Axiom A), and then further developed, following the influence of ideas of David Ruelle. In the smooth hyperbolic...
case, the SRB measure $\mu_\alpha$ corresponds to the fixed point of a transfer operator $L_\alpha$

enjoying a spectral gap on a suitable Banach space. In particular, this fixed point is a simple isolated eigenvalue in the spectrum of $L_\alpha$, and linear response can be viewed as an instance of perturbation theory for simple eigenvalues. This is evident in the linear response formulas, which all involve some avatar of the resolvent $(\id - L_\alpha)^{-1} = \sum_k L_\alpha^k$
applied to a suitable vector $Y_\alpha$, depending on the derivative of $\mu_\alpha$ and on $v_\alpha$.

It was soon realised that existence of a spectral gap is not sufficient to guarantee linear response when bifurcations are present (see e.g. [Ma,BI,BS]). In the other direction, neither the spectral gap nor structural stability is necessary for linear response, as was shown by Dolgopyat [Do] who obtained a linear response formula for some rapidly mixing systems (which were not all exponentially mixing or structurally stable).

The intuition that a key sufficient condition is convergence of the sum $\sum_k L_\alpha^k(Y_\alpha)$

was confirmed by [HM, Remark 2.4]. This is of course related to a summable decay of correlations. However, decay of correlation usually only holds for observables with a suitable modulus of continuity, which $Y_\alpha$, being a derivative, does not always enjoy. We confirm this intuition by studying a toy-model, of Pomeau-Manneville type: For $\alpha \in [0, 1)$, we consider the maps (as in [LSV]) $T_\alpha : [0, 1] \to [0, 1]$:

$$T_\alpha(x) = \begin{cases} 
  x(1 + 2^\alpha x^\alpha), & x \in [0, 1/2) \\
  2x - 1, & x \in [1/2, 1].
\end{cases}$$

(Of course, $T_0$ is just the angle-doubling map $T_0(x) = 2x$ modulo 1.) It is well-known that each such $T_\alpha$ admits a unique absolutely continuous invariant probability measure $\mu_\alpha = \rho_\alpha \, dx$. (Clearly, $\rho_0(x) \equiv 1$.) Statistical stability (continuity) of $\mu_\alpha$ when $\alpha$ changes is proved in [FT]. The absolutely continuous invariant probability measure $\mu_\alpha = \rho_\alpha \, dx$ is mixing for all $\alpha \in (0, 1)$. For $\alpha = 0$ the mixing rate for Lipschitz observables, say, is exponential (decaying like $1/2^k$). For $\alpha \in (0, 1)$ the mixing rate is only polynomial with rate $n^{1-1/\alpha}$ [Go,Sa]. (In fact, Gouëzel obtains a faster rate $n^{-1/\alpha}$ for $\int (\psi \circ T_\alpha^k) \, d\mu_\alpha$, if $\psi$ is bounded, $\phi$ is Lipschitz and vanishes in a neighbourhood of $x = 0$, and $\int \phi \, d\mu_\alpha = 0$, and this property is crucial below when $\alpha \geq 1/2$.) In particular, for any $\alpha \in (0, 1)$, the density $\rho_\alpha$ cannot be the fixed point of a transfer operator with a spectral gap on a Banach space containing all $C^\infty$ functions. However, we are able to prove (Theorem 2.1) that for any $q \in [1, \infty]$ and any $\psi \in L^q$, the map $\alpha \mapsto \int \psi \, d\mu_\alpha$
is continuously differentiable on $[0, 1 - 1/q)$, and we give two expressions [(2.6), with a resolvent, (2.7), of susceptibility function type] for the linear response formula, with $Y_\alpha = (X_\alpha N_\alpha(\rho_\alpha))'$, where $X_\alpha = v_\alpha \circ (T_\alpha^{-1}[0,1/2))$ and $N_\alpha$ corresponds to the first branch of the transfer operator $L_\alpha$. This is the first time that a linear response formula is achieved for a slowly mixing dynamics. The fact that linear response holds for any bounded $\psi$ is relevant since nonsmooth observables appear naturally. For example, if $A$ is smooth and $\Theta$ is a Heaviside function, the expectation value of $\Theta(A(x))$ gives the fraction of the total measure where $A$ has positive value, and more generally such discontinuous observables have probabilistic and physical interpretations, with the work of Lucarini et al. [Lu1,Lu2] showing how the theory of extremes for dynamical systems (in particular regarding climate change) can be cast in this framework.

Our proof is based on the cone techniques from [LSV], and hinges on the new observation that the factor $X_\alpha$, respectively $X_\alpha'$, compensates the singularity at zero of $\rho_\alpha$, respectively $\rho_\alpha$. Indeed, the compensation is drastic enough so that the $n^{-1/\alpha}$

\footnote{See Remark 2.2 for one possible generalisation.}