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Analytic Dependence is an Unnecessary Requirement in Renormalization of Locally Covariant QFT

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Abstract: Finite renormalization freedom in locally covariant quantum field theories on curved spacetime is known to be tightly constrained, under certain standard hypotheses, to the same terms as in flat spacetime up to finitely many curvature dependent terms. These hypotheses include, in particular, locality, covariance, scaling, microlocal regularity and continuous and analytic dependence on the metric and coupling parameters. The analytic dependence hypothesis is somewhat unnatural, because it requires that locally covariant observables (which are simultaneously defined on all spacetimes) depend continuously on an arbitrary metric, with the dependence strengthened to analytic on analytic metrics. Moreover the fact that analytic metrics are globally rigid makes the implementation of this requirement at the level of local $*$-algebras of observables rather technically cumbersome. We show that the conditions of locality, covariance, scaling and a naturally strengthened microlocal spectral condition, are actually sufficient to constrain the allowed finite renormalizations equally strongly, thus eliminating both the continuity and the somewhat unnatural analyticity hypotheses. The key step in the proof uses the Peetre–Slovák theorem on the characterization of (in general non-linear) differential operators by their locality and regularity properties.

1. Introduction

Perturbative ultraviolet renormalization of locally covariant quantum field theories in (globally hyperbolic) curved spacetime is a well established topic of algebraic quantum field theory, especially for scalar fields [5,6,13,14]. It essentially deals with two classes of objects: *Wick polynomials* and *time ordered Wick polynomials*. Exactly as in flat spacetime, these objects can be considered as the building blocks of the whole renormalization procedure. Smeared versions of Wick polynomials, of their time ordered products and of their derivatives generate an algebra $\mathcal{W}(M, \mathbf{g})$, for a given spacetime $(M, \mathbf{g})$, enlarged in a controlled way from the algebra of products of smeared linear fields. This enlarged algebra then includes physically fundamental observables, such as

the stress-energy tensor, which is necessary, for instance, to evaluate the energy densities and fluxes of physical processes in curved spacetimes like particle creation or Hawking radiation. The stress-energy tensor is also needed to compute the back reaction of the quantum matter on the background geometry.

This paper deals only with Wick polynomials, or more precisely just Wick powers with all results easily extended to all Wick polynomials by linearity, though the presented results could in principle be adapted to deal also with their derivatives and their time ordered products. In curved spacetime, Wick polynomials have to satisfy stronger locality and covariance requirements than in flat spacetime. These requirements are conveniently stated in the language of category theory introduced in [7], which we also use here. We should stress, though, that the categorical language primarily serves to compress somewhat long lists of hypotheses into concise statements. Existence of locally covariant Wick polynomials and their time ordered products was established in the seminal works of Hollands and Wald, respectively in [13,14]. It is well known that, in flat spacetime, time ordered Wick polynomials are not uniquely defined. This fact survives the passage to curved spacetime. However, unlike in flat spacetime, the absence of a preferred reference state means that Wick polynomials are *themselves* not uniquely defined. The ambiguities involved with the definition of these two classes of fields are physically interpreted as *finite renormalizations* or *renormalization counterterms*, upon adopting the natural locally covariant generalization of Epstein–Glaser approach to renormalization.

Exactly as in flat spacetime, each fixed type of (either Wick or time-ordered) polynomial admits a finite-dimensional class of independent counterterms. In curved spacetime, this class is much larger than in Minkowski space, because of the possible dependence of counterterms on *background curvature*. While this class may no longer be finite-dimensional, it is still *finitely generated* or *quasi-finite-dimensional* in a precise sense, because the counterterms may depend only polynomially on the curvature scalars up to a certain dimension. This remarkable result, in the case of Wick polynomials, presented in [13, Thm. 5.1] and summarized before the statement of our Theorem 3.1, is arrived at by imposing severe constraints on Wick polynomials in addition to those of locality and covariance. These requirements are of various kinds. Some arise from heuristic properties of quantum free fields, e.g., Hermiticity and commutation relations. Other requirements concern microlocal features which, loosely speaking, extend to curved spacetime the structure of Fourier transforms of the relevant Green functions on Minkowski space. Another requirement regards the behaviour of Wick polynomials under a rescaling of the metric and the parameters m^2 and ξ of the free theory, which describe the field's mass[1] and its coupling to the curvature. Finally there are the technically delicate requirements of *continuous* and *analytic* dependence on the metric. The two latter requirements play a crucial role in [13] in their proof of the strong restrictions on possible finite renormalization counterterms that was mentioned above.

The main difficulty with defining a suitable notion of the *continuous* dependence of an element of the algebra $\mathcal{W}(M, \mathbf{g})$ on the metric $\mathbf{g}$ (and the other parameters m^2 and ξ) is that, continuously changing the metric $\mathbf{g} \mapsto \mathbf{g}'$, the whole algebra $\mathcal{W}(M, \mathbf{g})$ changes correspondingly and algebras $\mathcal{W}(M, \mathbf{g})$ and $\mathcal{W}(M, \mathbf{g}')$ associated with different metrics are not canonically isomorphic. Therefore even just stating the condition of continuous dependence on $\mathbf{g}$ requires some finesse. Locality can be turned into an advantage in this context [13]. One may restrict attention to metric variations in a spacetime region $O \subset M$ with compact closure. If $\mathbf{g}$ agrees with $\mathbf{g}'$ outside O, essentially exploiting a

[1] As in [13], we will always treat m^2 as a real number, which could be either positive, zero, or even negative, as ultraviolet renormalization is not sensitive to the sign of m^2.