The Sum Over Topological Sectors and $\theta$ in the 2+1-Dimensional $\mathbb{C}P^1$ $\sigma$-Model

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Abstract: We discuss the three spacetime dimensional $\mathbb{C}P^N$ model and specialize to the $\mathbb{C}P^1$ model. Because of the Hopf map $\pi_3(\mathbb{C}P^1) = \mathbb{Z}$ one might try to couple the model to a periodic $\theta$ parameter. However, we argue that only the values $\theta = 0$ and $\theta = \pi$ are consistent. For these values the Skyrmions in the model are bosons and fermions respectively, rather than being anyons. We also extend the model by coupling it to a topological quantum field theory, such that the Skyrmions are anyons. We use techniques from geometry and topology to construct the $\theta = \pi$ theory on arbitrary 3-manifolds, and use recent results about invertible field theories to prove that no other values of $\theta$ satisfy the necessary locality.

1. Introduction

The functional integral definition of quantum field theory involves integrating over all possible configurations with a certain weight. It is often the case that the configuration space in the Euclidean functional integral breaks into topologically distinct sectors labeled by $\nu$. (These sectors and their characterization can depend on the Euclidean spacetime the theory is placed on.) Then, defining $Z_\nu$ as the sum over the configurations in the sector $\nu$, the total functional integral is given by a linear combination of $Z_\nu$

$$Z = \sum_\nu a_\nu Z_\nu. \quad (1.1)$$

The possible values of the coefficients $a_\nu$ are constrained by various consistency conditions like locality and unitarity. Different consistent choices of the $a_\nu$ correspond to distinct quantum field theories. An interesting problem is to find all possible consistent values of these coefficients, thus finding all possible theories constructed out of the building blocks $Z_\nu$.

A well known example is the quantum mechanical system of a single degree of freedom on a circle. Here, with Euclidean compact time the configuration space is the
space of maps $S^1 \to S^1$ and $\nu$ is the winding number. In this case the coefficients $a_\nu$ are constrained to be determined by a single periodic parameter $\theta$ as

$$a_\nu = e^{i\nu \theta}. \quad (1.2)$$

Another example is the 4$d$ pure $SU(N)$ gauge theory, where $\nu$ is the instanton number and again we have (1.2). In these two cases we can express $\nu$ as an integral of a local gauge invariant density and we can interpret (1.2) as arising from a term in the fundamental Lagrangian. In many situations $\nu$ cannot be written as an integral over a local density, but still an expression like (1.2) exists. A typical example is the 1 + 1-dimensional $SO(3)$-gauge theory, where $\nu$ is defined modulo 2 as the second Stiefel-Whitney class of a principal $SO(3)$-bundle, and correspondingly the allowed values of $\theta$ in (1.2) are 0 and $\pi$.

Locality and unitarity do not require $a_\nu$ to be the exponential of the integral of a local density, but rather they must be the partition functions of an invertible field theory [1]. In physics terms, $\log a_\nu$ can be thought of as an action of a classical field theory, which is local, but not necessarily an integral of a local density. Recent progress in understanding the structure of invertible theories can be brought to bear on the problem of combining $Z_\nu$ into a well-defined theory.

One of the goals of this paper is to clarify this sum over sectors in the 2+1 dimensional nonlinear $\mathbb{CP}^1$ $\sigma$-model. Placing the theory on $S^3$ and using the Hopf invariant, which is associated with $\pi_3(\mathbb{CP}^1) = \mathbb{Z}$, the label $\nu$ in (1.1) runs over the integers. It labels an instanton number. Then one might think that (1.2) is a consistent prescription for how to sum over these sectors and the theory is labeled by a continuous periodic parameter $\theta$. Explicitly, let $\vec{n}^2 = 1$ be a coordinate on $\mathbb{CP}^1 \simeq S^2$. Define $\text{Hopf}(\vec{n})$ to be a density such that $\int d^3x \, \text{Hopf}(\vec{n}) \in \mathbb{Z}$ is the Hopf invariant. Then, we can modify the standard Euclidean Lagrangian for $\vec{n}$ by adding a theta term (see e.g. [2,3] and many followup papers where this term was discussed) as follows

$$L = \frac{f}{2} (\partial \vec{n})^2 + i \theta \text{Hopf}(\vec{n}), \quad (1.3)$$

with a dimensionful parameter $f$. In this presentation it would seem that any $\theta$ is allowed and only $\theta \mod 2\pi$ matters. A hint that something might be wrong with this $\theta$ term comes from the fact that $\text{Hopf}(\vec{n})$ does not have a local expression in terms of $\vec{n}$. Furthermore, it is unclear how to define this theta term on other three-manifolds. Indeed, it has been known that $\theta = 0, \pi$ naturally arise in simple situations but not the other values of $\theta$. See e.g. [4] and references therein.

We will prove that, in fact, only $\theta = 0$ and $\pi$ are consistent. Furthermore, we will explicitly construct the corresponding mod 2 invariant on arbitrary spin three-manifolds. We will also present variants of the $\mathbb{CP}^1$ model, where the low-energy $\mathbb{CP}^1$ Goldstone bosons are coupled to a nontrivial TQFT leading to additional long range interactions such that $\theta$ behaves as if it has other values. These other values of $\theta$ are now allowed because we have modified the theory in the deep infrared. In condensed matter language, we could, for example, think about that as coupling the $\mathbb{CP}^1$ theory to a fractional quantum hall state.

1 The authors of [5] noted that certain microscopic 2+1 dimensional models of spins lead only to the values $\theta = 0$ and $\theta = \pi$. The same is true in 1+1 dimensions for such microscopic models. But unlike our claimed result in 2+1 dimensions, in 1+1 dimensions the $\mathbb{CP}^1$ model is well defined with arbitrary $\theta$ and not just at $\theta = 0, \pi$.

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