Commun. Math. Phys. 212, 591 – 611 (2000)

Communications in
Mathematical Physics

A Path Integral Approach to the Kontsevich Quantization Formula

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Received: 10 March 1999 / Accepted: 30 January 2000

Abstract: We give a quantum field theory interpretation of Kontsevich's deformation quantization formula for Poisson manifolds. We show that it is given by the perturbative expansion of the path integral of a simple topological bosonic open string theory. Its Batalin–Vilkovisky quantization yields a superconformal field theory. The associativity of the star product, and more generally the formality conjecture can then be understood by field theory methods. As an application, we compute the center of the deformed algebra in terms of the center of the Poisson algebra.

1. Introduction

In a recent paper [K], M. Kontsevich gave a general formula for the deformation quantization [BFFLS] of the algebra of functions on a Poisson manifold. The deformed product (the "star product") is given in terms of an expansion reminiscent of the Feynman perturbation expansion of a two dimensional field theory on a disc with boundary. We review Kontsevich's formula in Sect. 2.

The purpose of this paper is to describe this quantum field theory explicitly. It turns out that it is a simple bosonic topological quantum field theory on a disc D with a field $X : D \to M$ taking values in the Poisson manifold M and a one-form η on D taking values in the pull-back $X^*(T^*M)$ of the cotangent bundle. The formula for the star product is

$$f \star g\,(x) = \int_{X(\infty)=x} f(X(1))g(X(0))e^{\frac{i}{\hbar}S[X,\eta]}dX\,d\eta,$$

where $0, 1, \infty$ are three distinct points on the boundary of D. The integral is normalized in such a way that in the case of the trivial Poisson structure the star product reduces to the ordinary product. The action S is described in Sect. 3 and was originally studied for manifolds without boundary in [I] and [SchStr]. In particular the canonical quantization on the cylinder was considered.

In the symplectic case the above formula essentially reduces to the original Feynman path integral formula for quantum mechanics, as pointed out to us by H. Ooguri.

The quantization of the theory is somewhat subtle, due to the presence of a gauge symmetry which only closes on shell, as already noticed in [I]. In other words, the action S is a function of the fields annihilated by a distribution of vector fields which is only integrable on the set of critical points of S. As a consequence, the BRST quantization fails and one has to resort to the Batalin–Vilkovisky method (see for example [BV, W1, S1, AKSZ]).

This method yields a gauge fixed action, which turns out to have a superconformal invariance. Its perturbative expansion around constant classical solutions reproduces Kontsevich's formula.

As an application, we show in Sect. 4 by quantum field theory methods that there exists a star product equivalent to Kontsevich's whose center consists of the power series in $\hbar$ whose coefficients are in the center of the Poisson algebra. A rigorous proof of this statement will appear elsewhere [CFT].

More generally, we may consider a path integral associated to an arbitrary *polyvector field*, a formal sum of skew-symmetric contravariant tensor fields of arbitrary rank, the star product being the special case of bivector fields. Correlation functions of boundary fields yield then a map U from polyvector fields to polydifferential operators. Formal properties of this map can be deduced from BV and factorization methods of quantum field theory. This leads to identities, also found by Kontsevich, which may be thought of as the open string analog of the WDVV equations [W2, DDV]. They may be formulated by saying that U is an L_∞ morphism [SchlSt, LS]. They imply the associativity of the star product and, in the general setting of arbitrary polyvector fields, the formality conjecture [K]. These constructions are explained in Sect. 5.

Although the non-rigorous quantum field theory arguments of this paper are of course no substitute for the proofs in [K], this approach offers an explanation for why Kontsevich's construction works, and puts it in the context of Feynman's original picture of quantization [F]. Moreover, our approach indicates the way for more general constructions. In particular, one can consider the perturbative expansion around a non-trivial classical solution, one can insert a Hamiltonian and one can consider this quantum field theory on a complex curve of higher genus. We plan to study these variants in the future.

2. The Kontsevich Formula

In [K], M. Kontsevich wrote a beautiful explicit solution to the problem of deformation quantization of the algebra of functions on a Poisson manifold M. The problem is to find a deformation of the product on the algebra of smooth functions on a Poisson manifold, which to first order in Planck's constant is given by the Poisson bracket.

If M is an open set in $\mathbb{R}^d$ with a Poisson structure

$$\{f, g\}(x) = \sum_{i,j=1}^{d} \alpha^{ij}(x)\partial_i f(x)\partial_j g(x)$$

given by a skew-symmetric bivector field α, obeying the Jacobi identity

$$\alpha^{il}\partial_l\alpha^{jk} + \alpha^{jl}\partial_l\alpha^{ki} + \alpha^{kl}\partial_l\alpha^{ij} = 0, \tag{1}$$