A Fleming–Viot Particle Representation of the Dirichlet Laplacian

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Abstract: We consider a model with a large number $N$ of particles which move according to independent Brownian motions. A particle which leaves a domain $D$ is killed; at the same time, a different particle splits into two particles. For large $N$, the particle distribution density converges to the normalized heat equation solution in $D$ with Dirichlet boundary conditions. The stationary distributions converge as $N \to \infty$ to the first eigenfunction of the Laplacian in $D$ with the same boundary conditions.

1. Introduction

Our article is closely related to a model studied by Burdzy, Hołyst, Ingerman and March (1996) using heuristic and numerical methods. Although we are far from proving conjectures stated in that article, the present paper seems to lay solid theoretical foundations for further research in this direction. The model is related to many known ideas in probability and physics – we review them in the Appendix (Sect. 3). We present the model and state our main results in this section. Section 2 is entirely devoted to proofs.

We will be concerned with a multiparticle process. The motion of an individual particle will be represented by Brownian motion in an open subset of $\mathbb{R}^d$. Probably all our results can be generalized to other processes. However, the present paper is motivated by the article of Burdzy, Holyst, Ingerman and March (1996) whose results are very specific to Brownian motion and so we will limit ourselves to this special case. We note that Theorem 1.1 below uses only the strong Markov property of the process representing a particle and the continuity of the density of the hitting time of a set. Theorem 1.2 is similarly easy to generalize. At the other extreme, the proof of Theorem 1.4 uses Brownian properties in an essential way and would be hard to generalize. It might be therefore of some interest to see if Theorem 1.4 holds for a large class of processes.

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Consider an open set $D \subset \mathbb{R}^d$ and an integer $N \geq 2$. Let $X_t = (X^1_t, X^2_t, \ldots, X^N_t)$ be a process with values in $D^N$ whose evolution can be described as follows. Suppose $X_0 = (x^1, x^2, \ldots, x^N) \in D^N$. The processes $X^1, X^2, \ldots, X^N$ evolve as independent Brownian motions until the first time $t_1$ when one of them, say, $X^j$ hits the boundary of $D$. At this time one of the remaining particles is chosen in a uniform way, say, $X^k$, and the process $X^j$ jumps at time $t_1$ to $X^k$. The processes $X^1_t, X^2_t, \ldots, X^N_t$ continue as independent Brownian motions after time $t_1$ until the first time $t_2 > t_1$ when one of them hits $\partial D$. At the time $t_2$, the particle which approaches the boundary jumps to the current location of a randomly (uniformly) chosen particle among the ones strictly inside $D$. The subsequent evolution of the process $X_t$ proceeds along the same lines.

Before we start to study properties of $X_t$, we have to check if the process is well defined. Since the distribution of the hitting time of $\partial D$ has a continuous density, only one particle can hit $\partial D$ at time $t_k$, for every $k$, a.s. However, the process $X_t$ can be defined for all $t \geq 0$ using the informal recipe given above only if $t_k \to \infty$ as $k \to \infty$. This is because there is no obvious way to continue the process $X_t$ after the time $t_\infty = \lim_{k \to \infty} t_k$ if $t_\infty < \infty$. Hence, the question of the finiteness of $t_\infty$ has a fundamental importance for our model.

**Theorem 1.1.** We have $\lim_{k \to \infty} t_k = \infty$ a.s.

Consider an open set $D$ which has more than one connected component. If at some time $t$ all processes $X^j_t$ belong to a single connected component of $D$ then they will obviously stay in the same component from then on. Will there be such a time $t$? The answer is yes, according to the theorem below, and so we could assume without loss of generality that $D$ is a connected set, especially in Theorem 1.4.

**Theorem 1.2.** With probability 1, there exists $t < t_\infty$ such that all processes $X^j_t$ belong to a single connected component of $D$ at time $t$.

Before we continue the presentation of our results, we will provide a slightly more formal description of the process $X_t$ than that at the beginning of the introduction. The fully rigorous definition would be a routine but tedious task and so it is left to the reader. One can show that given $(x^1, x^2, \ldots, x^N) \in D^N$, there exists a strong Markov process $X_t$, unique in the sense of distribution, with the following properties. The process starts from $X_0 = (x^1, x^2, \ldots, x^N)$, a.s. Let

$$t_1 = \inf_{1 \leq m \leq N} \inf \{ t > 0 : \lim_{s \to t^-} X^m_s \in D^c \},$$

and for $n \geq 1$,

$$t_{n+1} = \inf_{1 \leq m \leq N} \inf \{ t > t_n : \lim_{s \to t^-} X^m_s \in D^c \}.$$

Then $t_{n+1} > t_n$ for every $n \geq 1$, a.s. For every $n \geq 1$, there exists a unique $k_n$ such that $\lim_{s \to t_n} X^m_s = X^m_{t_n}$ in $D^c$, a.s. We have $X^m_{t_n} = X^m_{t_n -}$, for every $m \neq k_n$. For some random $j = j(n, k_n) \neq k_n$ we have $X^j_{t_n} = X^j_{t_n -}$. The distribution of $j(n, k_n)$ is uniform on the set $\{1, 2, \ldots, N\} \setminus \{k_n\}$ and independent of $(X_t, 0 \leq t < t_n)$. For every $n$, the process $(X_{t \wedge t_n})_{t \geq t_n}$ is a Brownian motion on $D^N$ stopped at the hitting time of $\partial D^N$.

Let $P^D_t(x, dy)$ be the transition probability for the Brownian motion killed at the time of hitting of $D^c$. Given a probability measure $\mu_0(dx)$ on $D$, we define measures $\mu_t$ for $t > 0$ by

$$\mu_t(A) = \frac{\int_D P^D_t(x, A) \mu_0(dx)}{\int_D P^D_t(x, D) \mu_0(dx)}, \quad (1.1)$$