Commun. Math. Phys. 195, 549 – 583 (1998)

Communications in
Mathematical Physics

Formal GNS Construction and States in Deformation Quantization

Martin Bordemann, Stefan Waldmann

Fakultät für Physik, Universität Freiburg, Hermann-Herder-Str. 3, 79104 Freiburg i. Br., Germany.
E-mail: Martin.Bordemann@physik.uni-freiburg.de; Stefan.Waldmann@physik.uni-freiburg.de

Received: 14 August 1996 / Accepted: 15 December 1997

Abstract: In this paper we develop a method of constructing Hilbert spaces and the representation of the formal algebra of quantum observables in deformation quantization which is an analog of the well-known GNS construction for complex C^*-algebras: in this approach the corresponding positive linear functionals ("states") take their values not in the field of complex numbers, but in (a suitable extension field of) the field of formal complex Laurent series in the formal parameter. By using the algebraic and topological properties of these fields we prove that this construction makes sense and show in physical examples that standard representations such as the Bargmann and Schrödinger representation come out correctly, both formally and in a suitable convergence scheme. For certain Hamiltonian functions (contained in the Gel'fand ideal of the positive functional) a formal solution to the time-dependent Schrödinger equation is shown to exist. Moreover, we show that for every Kähler manifold equipped with the Fedosov star product of Wick type all the classical delta functionals are positive and give rise to some formal Bargmann representation of the deformed algebra.

1. Introduction

In the programme of deformation quantization introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [6] the algebra of quantum observables is considered as an associative local formal deformation (a so-called star product $*$) of the associative commutative algebra of smooth complex-valued functions $C^\infty(M)$ on a given symplectic manifold M, such that the first order commutator equals $i\lambda$ times the Poisson bracket and such that complex conjugation is an antilinear involution of the deformed algebra. The latter is equal to $C^\infty(M)[[\lambda]]$, the space of formal power series in the deformation parameter λ with coefficients in $C^\infty(M)$, and the associative noncommutative multiplication $*$ is bilinear with respect to the ring $\mathbb{C}[[\lambda]]$ of formal power series in λ with complex coefficients. On one hand the rather difficult question of existence

of these deformations for general symplectic manifolds has positively been answered (DeWilde-Lecomte 1983 [18]; Fedosov 1985 [21, 22]; Omori-Maeda-Yoshioka 1991 [41]). Moreover their classification (up to equivalence) in terms of formal power series with coefficients in the second de Rham cohomology class of the underlying symplectic manifold has recently been achieved ([8, 40, p. 204]).

On the other hand star products have the –at first sight rather unpleasant– feature of lacking convergence uniform in the formal parameter which is due to the fact that they depend on the infinite jets of the two functions which in turn can be made as divergent as possible by Borel's Theorem (see e.g. [49]; see also the article by Rubio 1984 [44] for the commutativity of local associative products on $C^\infty(M)$). Moreover, the deformed algebras do not seem to have obvious representations in some complex separable Hilbert space which is unsatisfactory from the physical point of view. In the past decade, however, several people have attacked this problem: Cahen, Gutt, and Rawnsley [14, 15] start from the finite-dimensional operator algebras of geometric quantization in tensor powers of a very ample regular prequantum line bundle over a compact Kähler manifold and use coherent states (see [7, 43]) to first construct star products for the Berezin-Rawnsley symbols ([7, 43]) for each tensor power separately. In a second step an asymptotic expansion of these star products in the inverse tensor power is shown to define a local star product on the manifold where the formal parameter appears as a sort of interpolation of the inverse tensor powers. For compact Hermitian symmetric spaces they showed that the subspace of representative functions has a convergent star product. See also [9, 10] for an elementary algebraic approach in the particular case of complex projective space. Pflaum has studied star products and their convergence on suitable subspaces on cotangent bundles in his thesis [42]. In flat $\mathbb{R}^{2n}$ a non-formal analogue ("twisted products") of star products using integral formulas can be defined on the Schwartz test function space (which is thereby made into an associative topological complex algebra) and extended to larger spaces by functional analytic techniques. The usual Weyl-Moyal star product is recovered as a asymptotic expansion in $\hbar$ (which can be introduced as an dilatation parameter), see [30, 34, 37] for more details. Moreover, Fedosov has shown that the deformed algebra allows a so-called asymptotic operator representation in a complex Hilbert space if and only if certain integrality properties of a formal index are satisfied which he defines as a formal analogue of the Atiyah-Singer index theorem and which is an invariant of the underlying symplectic manifold (see e. g. Fedosov's book [22] for a detailed exposition).

The approach of this paper is motivated by the following consideration: in the theory of complex C^*-algebras (which form one of the main mathematical pillars of algebraic quantum field theory (see Haag's book [29] for details) and —in a more algebraic context— of Connes' noncommutative geometry (see Connes' book [16])) the representing complex Hilbert spaces for a given complex C^*-algebra A are constructed by means of the so-called GNS representation (see e. g. [13]): roughly speaking, any positive complex-valued linear functional on A (i.e. which maps positive elements of A to nonnegative real numbers) gives rise to a left ideal I of A (the Gel'fand ideal), and A is canonically represented on the quotient space A modulo I. Due to the positivity of the initial functional this quotient is equipped with a positive definite sesquilinear form and thus becomes a complex pre-Hilbert space whose completion yields the desired representation of the algebra. The initial functional can be regarded as a vacuum expectation value (functional) or a *state* on A if A has a unit element.

A natural question which has been brought to our attention by K. Fredenhagen is the following: can the GNS construction be extended to the associative algebras occurring in deformation quantization? At first sight, this seems almost impossible: firstly, for any