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On the Algebras of BPS States

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Abstract: We define an algebra on the space of BPS states in theories with extended supersymmetry. We show that the algebra of perturbative BPS states in toroidal compactification of the heterotic string is closely related to a generalized Kac–Moody algebra. We use D-brane theory to compare the formulation of RR-charged BPS algebras in type II compactification with the requirements of string/string duality and find that the RR charged BPS states should be regarded as cohomology classes on moduli spaces of coherent sheaves. The equivalence of the algebra of BPS states in heterotic/IIA dual pairs elucidates certain results and conjectures of Nakajima and Gritsenko & Nikulin, on geometrically defined algebras and furthermore suggests nontrivial generalizations of these algebras. In particular, to any Calabi–Yau 3-fold there are two canonically associated algebras exchanged by mirror symmetry.

1. Introduction

String theories and field theories with extended supersymmetry have a distinguished set of states in their Hilbert space known as BPS states. Thanks to supersymmetry, one can make exact statements about these magical states even in the face of all the complexities, perplexities, and uncertainties that plague most attempts to understand nonperturbative Quantum Field Theory and Quantum String Theory. As a result, they have played a special role in the study of strong-weak coupling duality in both field theory and string theory.

In this paper we point out that there is a simple, physical, and universal property of BPS states: They form an algebra. There are four reasons the algebra of BPS states is interesting:

1. BPS algebras appear to be infinite-dimensional gauge algebras, typically spontaneously broken down to a finite dimensional unbroken gauge symmetry.

2. Comparing BPS algebras in dual string pairs has important applications in mathematics.
3. The BPS algebras appear to control the threshold corrections in $d = 4, \mathcal{N} = 2$ string compactification.
4. BPS algebras appear to be intimately related to black hole physics. In particular, the counting of nonperturbative black hole degeneracies seems to be related to generalized Kac–Moody algebras.

We will discuss (1) and (2) in this paper. Item (3) is the subject of several papers [1–5]. Item (4) has been proposed recently in an imaginative paper of Dijkgraaf, Verlinde, and Verlinde [6].

The outline of this paper is as follows. Section Two contains the basic definition of the algebra of BPS states. In Section Three we use the definition to compute the algebra of perturbative BPS states in toroidal compactifications of heterotic string theory and discuss the relation of this algebra to Generalized Kac–Moody (GKM) algebras.[1] In the fourth section we turn to an analysis of BPS states in Type II string theory, we discuss the formulation in terms of moduli spaces and argue that sheaves provide the correct language for a general discussion of BPS states. Section 5 develops the sheaf-theoretic interpretation of BPS states in some detail for $K3$ and T^4 compactifications. In Sect. 6 this is extended to the Calabi–Yau case. Section 7 contains a conjectural method for the computation of the algebra of BPS states in Type II string theory. String duality predicts isomorphisms between certain algebras of Type II BPS states and the dual algebra of perturbative heterotic BPS states. In Sects. 8 and 9 we discuss this isomorphism in a certain limit. The final section contains brief conclusions and a discussion of open issues.

2. The Space of BPS States is Always an Algebra

2.1. Definition. The definition of the algebra of BPS states uses very little information and is therefore quite general. We suppose that

1. There are absolutely conserved charges Q and therefore the Hilbert space of asymptotic particle states is graded $\mathcal{H} = \oplus \mathcal{H}^Q$.
2. In each superselection sector there is a Bogomolnyi bound on the energy:

$$E \geq \parallel \mathcal{Z}(Q) \parallel, \tag{2.0}$$

where $\mathcal{Z}(Q)$ is a central charge and $\parallel \cdot \parallel$ is some norm function.

Given the two conditions above we can define the Hilbert space of BPS states $\mathcal{H}_{BPS}$ to be the space of *one-particle* states saturating (2.0).[2] An algebra is simply a vector space with a product and we can define the product

$$\mathcal{R} : \mathcal{H}_{BPS} \otimes \mathcal{H}_{BPS} \rightarrow \mathcal{H}_{BPS} \tag{2.1}$$

as follows. Take two BPS states ψ_i of charges Q_i, $i = 1, 2$. Boost them by momenta $\pm \vec{p}_*$ to produce a two-body state in the center of mass frame such that the total energy satisfies the BPS bound $E = \parallel \mathcal{Z}(Q_1 + Q_2) \parallel$. By definition $\mathcal{R}(\psi_1 \otimes \psi_2)$ *is the orthogonal projection of* $\Lambda_{\vec{p}_*}(\psi_1) \otimes \Lambda_{-\vec{p}_*}(\psi_2)$ *onto* $\mathcal{H}_{BPS}^{Q_1+Q_2}$, *where* $\Lambda_{\vec{p}}$ *is the Lorentz boost.* The algebra (2.1) is the central object of study in this paper.

[1] In this paper the term GKM is used for something slightly different from the object defined by Borcherds. See note added.

[2] If the one-particle state is a bound state at threshold it can be distinguished from a two-particle state satisfying (2.0) by the representation of the supertranslation group.