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Hopf Algebras, Cyclic Cohomology and the Transverse Index Theorem

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Abstract: In this paper we solve a longstanding internal problem of noncommutative geometry, namely the computation of the index of transversally elliptic operators on foliations. We show that the computation of the local index formula for transversally hypoelliptic operators can be settled thanks to a very specific Hopf algebra $\mathcal{H}_n$, associated to each integer codimension. This Hopf algebra reduces transverse geometry, to a universal geometry of affine nature. The structure of this Hopf algebra, its relation with the Lie algebra of formal vector fields as well as the computation of its cyclic cohomology are done in the present paper, in which we also show that under a suitable unimodularity condition the cosimplicial space underlying the Hochschild cohomology of a Hopf algebra carries a highly nontrivial cyclic structure.

Introduction

In this paper we present the solution of a longstanding internal problem of noncommutative geometry, namely the computation of the index of transversally elliptic operators on foliations.

The spaces of leaves of foliations are basic examples of noncommutative spaces and already exhibit most of the features of the general theory. The index problem for longitudinal elliptic operators cf. [Co, M-S] is simple to formulate in the presence of a transverse measure and leads in general to the construction ([C-S]) of a natural map from the geometric group to the K theory of the leaf space, i.e. the K theory of the associated C^* algebra. This "assembly map" μ is known in many cases to exhaust the K theory of the C^* algebra but property T in the group context and its analogue for foliations give an obstruction to tentative proofs of its surjectivity in general. One way to test the K group, $K(C^*(V,F)) = K(V/F)$ for short, is to use its natural pairing with the *K-homology* group of $C^*(V,F)$. Cycles in the latter represent "abstract elliptic operators" on V/F and the explicit construction for *general* foliations of such cycles is

already a quite elaborate problem. The point is that we do not want to assume any special property of the foliation such as, for instance, the existence of a holonomy invariant transverse metric as in Riemannian foliations. Equivalently, we do not want to restrict in any way the holonomy pseudogroup of the foliation.

In [Co1, H-S, C-M], a general solution was given to the construction of transversal elliptic operators for foliations. The first step ([Co1]) consists in passing by a Thom isomorphism to the total space of the bundle of transversal metrics. This first step is a geometric adaptation of the reduction of an arbitrary factor of type III to a crossed product of a factor of type II by a one parameter group of automorphisms. Instead of only taking care of the volume distorsion (as in the factor case) of the involved elements of the pseudogroup, it takes care of their full Jacobian. The second step ([H-S]) consisted in realizing that while the standard theory of elliptic pseudodifferential operators cannot be used to construct the desired K-homology cycle, it suffices to replace it by its refinement to hypoelliptic operators. This was used in [C-M] in order to construct a *differential* (hypoelliptic) operator D solving the general construction of the K-cycle.

One then disposes of a well posed general index problem. The index defines a map: $K(V/F) \to \mathbb{Z}$ which is simple to compute for those elements of $K(V/F)$ in the range of the assembly map. The problem is to provide a general formula for the cyclic cocycle $\mathrm{ch}_*(D)$ which computes the index by the equality,

$$\langle \mathrm{ch}_*(D), \mathrm{ch}^*(E) \rangle = \mathrm{Index}\, D_E \qquad \forall\, E \in K(V/F), \tag{1}$$

where the chern character $\mathrm{ch}^*(E)$ belongs to the cyclic homology of V/F. We showed in [C-M] that the spectral triple given by the algebra $\mathcal{A}$ of the foliation, together with the operator D in Hilbert space $\mathcal{H}$ actually fulfills the hypothesis of a general abstract index theorem, holding at the operator theoretic level. It gives a "local" formula for the cyclic cocycle $\mathrm{ch}_*(D)$ in terms of residues extending the ideas of the Wodzicki–Guillemin–Manin residue and the Dixmier trace. Adopting the notation $\displaystyle{-\!\!\!\!\!\!\int}$ for such a residue the general formula gives the components φ_n of the cyclic cocycle $\varphi = \mathrm{ch}_*(D)$ as universal *finite* linear combinations of expressions which have the following general form,

$$-\!\!\!\!\!\!\int a^0\, [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)}\, |D|^{-n-2|k|} \qquad \forall\, a^j \in \mathcal{A}, \tag{2}$$

where for an operator T in $\mathcal{H}$ the symbol $T^{(k)}$ means the k^{th} iterated commutator of D^2 with T.

It was soon realized that though the general index formula easily reduces to the local form of the Atiyah-Singer index theorem when D is say a Dirac operator on a manifold, the actual explicit computation of all the terms (2) involved in the cocycle $\mathrm{ch}_*(D)$ is a rather formidable task. As an instance of this let us mention that even in the case of codimension one foliations, the printed form of the explicit computation of the cocycle takes around one hundred pages. Each step in the computation is straightforward but the explicit computation for higher values of n is clearly impossible without a new organizing principle which allows one to bypass them.

In this paper we shall adapt and develop the theory of cyclic cohomology to the relevant class of Hopf algebras and show that this provides exactly the missing organizing principle, thus allowing to perform the computation for arbitrary values of n. We shall construct for each value of n a specific Hopf algebra $\mathcal{H}(n)$, show that it acts on the C^* algebra of the transverse frame bundle of any codimension n foliation (V, F) and that the index computation takes place within the cyclic cohomology of $\mathcal{H}(n)$. We compute this