Universal Construction of $\mathcal{W}_{q,p}$ Algebras

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Abstract: We present a direct construction of the abstract generators for $q$-deformed $\mathcal{W}_N$ algebras. New quantum algebraic structures of $\mathcal{W}_{q,p}$ type are thus obtained. This procedure hinges upon a twisted trace formula for the elliptic algebra $A_{q,p}(s\hat{l}(N)_c)$ generalizing the previously known formulae for quantum groups. It represents the $q$-deformation of the construction of $\mathcal{W}_N$ algebras from Lie algebras.

1. Introduction

The connection between $q$-deformed Virasoro (more generally $q$-deformed $\mathcal{W}$ algebras), and elliptic quantum $A_{q,p}(s\hat{l}(2)_c)$ (resp. $A_{q,p}(s\hat{l}(N)_c)$) algebras, was investigated in our recent papers [1–4]. It was shown that $q$-deformed Virasoro and $\mathcal{W}$ structures [5–7] were present inside the $A_{q,p}(s\hat{l}(N)_c)$ elliptic algebra [8, 9]; at the quantum level, once a particular relation existed between the central charge $c$, the elliptic nome $p$ and the deformation parameter $q$: $(-p^2)^NM = q^{-c-N}$ for some integer $M$, and at the classical limit, obtained when setting an additional relation $p = q^{Nh}$ for some integer $h$. This classical limit was identified in particular cases with the classical $q$-deformed algebras constructed in [5]. It also yielded different classical deformed algebras. In this way, one obtained directly a set of quantizations of these classical $q$-deformed Poisson algebras, in particular from [5], interestingly distinct from the original quantization [7] obtained from explicit bosonic realizations.

The construction was achieved at the abstract level in that only the abstract algebraic relations for $A_{q,p}(s\hat{l}(N)_c)$, defined by the eight vertex model $R$-matrix [10], were used to derive the $q$-deformed structures. It was assumed throughout the derivations that the initial formal series relations [8] were in fact extended to the level of analytic relations, thereby leading from one single exchange relation for this generating operator functional of the algebras to an infinite $\mathbb{Z}$-labeled set of exchange relations for the modes, depending
upon the choice of a relevant series expansion in a crown-shaped sector for the ratio of spectral parameters in the elliptic structure function.

In our initial approach [3], the extension of the construction to $sl(N)$ was achieved by defining the abstract generators of higher spin simply as shifted ordered products of the spin one generators $t(z) = \text{Tr} \left[ L^*(zq^{c/2})(L^-(z))^{-1} \right]$. In this respect, the first derivation cannot be considered as the $q$-deformed version of the $W_N$ algebra construction [11] which takes as generators combinations of the current algebra generators from which one then extracts $sl(N)$ scalar objects; the detailed study developed in [11] allows us to appreciate the successes and the difficulties of this approach. The construction [3] however gave rise to perfectly consistent non-trivial algebraic structures. Indeed the shift in the spectral parameters when defining the product of spin-one generators precludes any identification of the subsequent $W_{q,p}$ algebras as subsets of the enveloping algebras of any particular integer-labeled $Vir_q(sl(2))$ derived from the spin-one generators within a given choice of sector for the ratio of spectral parameters. In particular, as indicated before, the classical limit of these algebras $W_{q,p}$ did lead in given cases to the original [5] classical $q$-$VW$ Poisson algebra, characterizing the quantum structure as a genuine $q$-deformed $W_N$ algebra.

The question of a universal construction of $W_{q,p}$ algebras, which would not give any privileged role to the spin-one operators as "building blocks" of the full algebra, thus remained open. Our purpose here is to present such a construction and compute the related $q$-deformed algebraic structures. With this intention we shall rely upon basic algebraic structures derived from the properties of the operator $z \equiv L^*(q^2 z)^2 (L^- (z))^{-1}$ which was the fundamental object in our previous derivation.

In a first section we recall the main results obtained in [3], and the main notations and definitions concerning the elliptic algebras.

In a second section, we prove that $z$ obeys an exchange relation of the type $R z R' \equiv R' z R \in R \otimes R$. Originally derived and discussed in [13, 14], these exchange relations then lead us to define new surfaces in the $(p, q, c)$ space on which quantum, then classical, $q$-Virasoro algebras of the same type as in [1] arise.

The classical structures are the same as in [1]. The quantum structures by contrast are more general, for $N \geq 3$, than the original algebras derived in [3], which one recovers as particular cases. One cannot however directly derive higher order generators from such an exchange algebra, contrary to the simpler case $RLL = LLR$, where a famous twisted trace formula exists [15, 16] to generate quantum commuting Hamiltonians. But since the definition of the basic elliptic algebra involves two distinct $R$-matrices as $RLL = LLR^*$, one cannot apply [15, 16] to it either.

In a third, central section we show how to define a suitable twisted trace formula, involving $z \equiv L^*(q^2 z)^2 (L^- (z))^{-1}$ and the $R$-matrix of the algebra $A_{q,p}(sl(N)_c)$.

This formula now provides us with the sought-for $q$-deformation of the canonical procedure [11] for $W_N$ algebras. It leads to closed exchange algebras of the quantum $W_{q,p}$ type, when the generalized relation $(-p^2)^n = q^{-c-N}$ is obeyed. Here $n$ is any integer, not necessarily a multiple of $N$. One then gets classical $q$-$VW$ Poisson algebras when $p = q^N$. The quantum and classical algebras $W_{q,p}[sl(N)]$ thus constructed contain, when $n = kN$ for integer $k$, the structures obtained in [3] by using the shifted products.

As emphasized, we now have a really universal construction of the higher $q$-deformed currents generating these $W_{q,p}$ algebras. In addition, we have obtained the elliptic algebra version of the twisted trace construction, bypassing thereby the difficulty generated a