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Quantum Monodromy in Integrable Systems

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Abstract: Let $P_1(h), \dots, P_n(h)$ be a set of commuting self-adjoint h-pseudo-differential operators on an n-dimensional manifold. If the joint principal symbol p is proper, it is known from the work of Colin de Verdière [6] and Charbonnel [3] that in a neighbourhood of any regular value of p, the joint spectrum locally has the structure of an affine integral lattice. This leads to the construction of a natural invariant of the spectrum, called the quantum monodromy. We present this construction here, and show that this invariant is given by the classical monodromy of the underlying Liouville integrable system, as introduced by Duistermaat [9]. The most striking application of this result is that all two degree of freedom quantum integrable systems with a *focus-focus* singularity have the same non-trivial quantum monodromy. For instance, this proves a conjecture of Cushman and Duistermaat [7] concerning the quantum spherical pendulum.

1. Introduction

Obstructions to the existence of global action-angle coordinates for completely integrable systems are well known since Duistermaat's article [9]. It was then natural to raise the question about the impact of these obstructions on *quantum* integrable systems, at least for the (semi)-classical pseudo-differential quantisation on cotangent bundles. The first attempts in this direction were [7] and [11], both of them concerning the monodromy invariant for the example of the spherical pendulum. This system is indeed one of the simplest (along with the Champagne bottle [1]) that exhibits a non-trivial monodromy. The first of these articles [7] proposed a particularly interesting way of detecting the monodromy by observing a shift in the lattice structure of the joint spectrum. It is the purpose of this article to state, prove and explain this idea.

Surprisingly enough, this idea of quantum monodromy has been sleeping for ten years, before new interest resulted in its experimental discovery in the spectrum of excited water molecules [4,5].

Back to mathematics, it turns out that, in the framework of semi-classical microlocal analysis (developed for integrable systems in [3]), there is a natural way of defining an invariant of the joint spectrum away from singularities of the principal symbols, that precisely describes the obstruction to the existence of a *global* lattice structure for the spectrum. The organisation of this article is as follows: we first extract the relevant properties of joint spectra, and define the *quantum monodromy* invariant for any set that shares these properties (Sect. 2). Then we prove in Sect. 3 that, for spectra, the quantum monodromy is precisely given by the classical monodromy of the underlying classical Hamiltonian system. The result is applied in Sect. 4 to the particularly interesting case of systems admitting a *focus-focus* singularity. The last Sect. 5 finally shows how to read off the monodromy from a picture of the spectrum. As an example, we use the spectrum of the Champagne bottle computed by Child [4].

2. Construction of the Quantum Monodromy

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^n$, let H be a set of positive real numbers accumulating at 0, and for any h in H let $\Sigma(h)$ be a discrete subset of $\mathcal{U}$.

If B is an open subset of $\mathcal{U}$, a family $(f(h))_{h\in H}$ of smooth functions on B with values in $\mathbb{R}^n$ is called a *symbol* (of order zero) if it admits an asymptotic expansion of the form

$$f(h) = f_0 + hf_1 + h^2 f_2 + \cdots$$

for smooth functions $f_i : B \to \mathbb{R}^n$. More precisely we require that for any $\ell \geq 0$, for any $N \geq 0$, and for any compact $K \subset B$, there is a constant $C_{\ell,N,K}$ such that for all $h \in H$,

$$\left\| f(h) - \sum_{k=0}^{N} h^k f_k \right\|_{\ell} \leq C_{\ell,N,K} h^{N+1},$$

where $\|.\|_\ell$ denotes the C^ℓ norm in K. The symbol $f(h)$ is *elliptic* if its principal part f_0 is a local diffeomorphism of B into $\mathbb{R}^n$. The value of $f(h)$ at a point $c \in B$ will be denoted by $f(h; c)$.

A family $(r(h))_{h\in H}$ of elements of a finite dimensional vector space is said to be $O(h^\infty)$ if for any $N \geq 0$ there is a constant $C > 0$ such that $\|r(h)\| \leq Ch^N$, uniformly for all $h \in H$. If $S(h)$ is any family of sets depending on h, then the notation $f(h) \in S(h) + O(h^\infty)$ means that the function $\mathrm{dist}(f(h), S(h))$ is $O(h^\infty)$.

We will say that $\Sigma(h)$ has the structure of an "asymptotic affine lattice" whenever it can be described with a locally finite set of "asymptotic affine integral charts", in the following sense:

Definition 1. *$(\Sigma(h), \mathcal{U})$ is an "asymptotic affine lattice" if for any $c \in \mathcal{U}$, there exists a small open ball $B \subset \mathcal{U}$ around c, and an elliptic symbol $f(h) : B \to \mathbb{R}^n$ of order zero such that, for any family $\lambda(h) \in B$:*

- $\lambda(h) \in \Sigma(h) \cap B + O(h^\infty) \iff f(h; \lambda(h)) \in h\mathbb{Z}^n + O(h^\infty)$
- *if $\lambda(h)$ and $\lambda'(h)$ are in $\Sigma(h) \cap B$, then $\lambda'(h) - \lambda(h) = O(h^\infty)$ if and only if for small h, $\lambda'(h) = \lambda(h)$.*