The Wulff Construction in Three and More Dimensions

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Abstract: In this paper we prove the Wulff construction in three and more dimensions for an Ising model with nearest neighbor interaction.

1. Introduction

The problem of phase separation for two dimensional Ising model and the study of the equilibrium shape of crystals (Wulff shape) has been initiated by Dobrushin, Kotecky and Shlosman [DKS]. Among other things, they proved that if at very low temperatures we decrease the averaged magnetization in the pure phase, we observe the creation of a macroscopic droplet of the phase which has a deterministic shape on the macroscopic scale.

The proof has been first simplified by Pfister [Pf] and then extended to the whole of the phase transition region by Ioffe [I1, I2] (see also [SS] and [PV]). Recently, Ioffe and Schonmann [IS] have completed the DKS theory up to the critical temperature and greatly simplified the original proofs. Moderate deviations in the exact canonical ensemble are also studied in [IS].

In two dimensions, the proofs have been based on duality arguments and on a coarse graining procedure (skeleton). These arguments do not seem to apply in higher dimensions. For more than two dimensions, an alternative procedure has been proposed by Alberti, Bellettini, Cassandro and Presutti [ABCP, BCP] for Ising systems with Kac potentials. They rephrase the whole problem in terms of $L^1$ theory and prove large deviations for the appearance of a droplet of the minority phase in a scaling limit when the size of the domain diverges not much faster than the range of the Kac potentials. This amounts to a weak large deviation principle which is obtained by proving $\Gamma$–convergence of a functional associated to the spins system [ABCP]. A large deviation principle has then been proved via a tightness property [BCP].
Their approach has been generalized by Benois, Bodineau, Butta and Presutti [BBBP, BBP] by taking first the thermodynamic limit and then letting the range of interaction go to infinity. The first paper [BBBP] was devoted to the proof of a weak large deviation principle for the macroscopic magnetization which is equivalent to the computation of surface tension. The main idea has been to introduce a coarse graining in order to use the $L^1$ setting. Namely, events in $L^1$ were related to mesoscopic quantities by an argument which we will refer to later as minimal section argument. An exact expression of surface tension was difficult to recover from coarse grained estimates and surface tension was only derived in the Kac limit, i.e. when the range of interactions tends to infinity. The second step [BBP] consisted of proving a tightness property by using the compactness in $L^1$ of the set of functions of bounded variation with finite perimeter.

Wulff construction for three dimensional independent percolation has been proven by Cerf [Ce] using a procedure similar to the one of [BBBP] and a novel definition of surface tension. In this case, the dependence on boundary conditions is weaker and, the minimal section argument enables to prove directly a weak large deviation principle by using this appropriate definition of surface tension. As percolation occurs in an infinite volume, there is an extra difficulty and different compactness arguments have been required.

In this paper we proceed as in [BBBP]. One of the main difficulties is to recover surface tension from a constraint on the averaged magnetization. The surface tension is defined as $\log \left( \frac{Z_C}{Z_{\bar{C}}} \right)$, where the partition functions are computed with $+$ boundary conditions and with mixed boundary conditions ($+$ at the top and $-$ at the bottom), see for instance the paper of Messager, Miracle-Solé and Ruiz [MMR]. To use directly this definition, one would have to find in the bulk surfaces of $+$ spins or of $-$ spins which in fact may not exist. A way to circumvent this problem is to prove that surface tension can be produced by averaging the boundary conditions, choosing the spins with respect to the $+$ pure phase and to the $-$ pure phase.

For the Ising model with nearest neighbor interaction, the coarse graining developed by Pisztora [Pi1] will play an analogous role to the one used for the Kac model. Pisztora’s coarse graining is one of the most profound and powerful techniques for the study of the Ising (Potts) model, it provides an accurate description of the Ising model in a non-perturbative regime up to a temperature $\hat{T}_c$ which is conjectured to agree with the critical temperature. In the following, we will mention which of our results hold up to $\hat{T}_c$. As Pisztora’s coarse graining is defined via the FK representation, several quantities need to be rewritten in terms of the FK representation. In particular, our approach to the surface tension (Sect. 4) is built upon the FK representation and, is motivated by the corresponding construction in [Ce]. This is a key to obtain precise surface order estimates on the logarithmic scale. This is also the only point at which we refer to [Ce], the core philosophy of our proof is based on the renormalization ideas of [BBBP] and [BBP], including the appropriate setup of the geometric measure theory. The coarse graining schemes of the latter works, however, depend on specific properties of Kac potentials and, one of our main technical tasks here is to develop a relevant modification of these renormalization procedures in the nearest neighbor context. A step further in the understanding of the surface tension will be to prove a phase separation theorem for the Kac model with finite range interactions by using a coarse graining defined only in terms of the Gibbs measure [Bo].

After introducing the main notation, we state in Sect. 2 the results and an overview of the paper (see Subsect. 2.3).