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Renormalization in Quantum Field Theory and the Riemann–Hilbert Problem I: The Hopf Algebra Structure of Graphs and the Main Theorem

Alain Connes, Dirk Kreimer

Institut des Hautes Etudes Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France.
E-mail: connes@ihes.fr; kreimer@ihes.fr

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Abstract: This paper gives a complete selfcontained proof of our result announced in [6] showing that renormalization in quantum field theory is a special instance of a general mathematical procedure of extraction of finite values based on the Riemann–Hilbert problem. We shall first show that for any quantum field theory, the combinatorics of Feynman graphs gives rise to a Hopf algebra $\mathcal{H}$ which is commutative as an algebra. It is the dual Hopf algebra of the enveloping algebra of a Lie algebra $\underline{G}$ whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of $\mathcal{H}$. We shall then show that, using dimensional regularization, the bare (unrenormalized) theory gives rise to a loop

$$\gamma(z) \in G\,, \qquad z \in C,$$

where C is a small circle of complex dimensions around the integer dimension D of space-time. Our main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of γ. We begin to analyse the group G and show that it is a semi-direct product of an easily understood abelian group by a highly non-trivial group closely tied up with groups of diffeomorphisms. The analysis of this latter group as well as the interpretation of the renormalization group and of anomalous dimensions are the content of our second paper with the same overall title.

1. Introduction

This paper gives a complete selfcontained proof of our result announced in [6] showing that renormalization in quantum field theory is a special instance of a general mathematical procedure of extraction of finite values based on the Riemann–Hilbert problem. In order that the paper be readable by non-specialists we shall begin by giving a short

introduction to both topics of renormalization and of the Riemann–Hilbert problem, with our apologies to specialists in both camps for recalling well-known material.

Perturbative renormalization is by far the most successful technique for computing physical quantities in quantum field theory. It is well known for instance that it accurately predicts the first ten decimal places of the anomalous magnetic moment of the electron.

The physical motivation behind the renormalization technique is quite clear and goes back to the concept of effective mass in nineteenth century hydrodynamics. Thus for instance when applying Newton's law,

$$F = m\, a, \tag{1}$$

to the motion of a spherical rigid balloon B, the inertial mass m is not the mass m_0 of B but is modified to

$$m = m_0 + \tfrac{1}{2}\, M, \tag{2}$$

where M is the mass of the volume of air occupied by B. It follows for instance that the initial acceleration a of B is given, using the Archimedean law, by

$$-(M - m_0)g = \left(m_0 + \tfrac{1}{2}\, M\right) a \tag{3}$$

and is always of magnitude less than $2g$. The additional inertial mass $\delta\, m = m - m_0$ is due to the interaction of B with the surrounding field of air and if this interaction could not be turned off there would be no way to measure the mass m_0.

The analogy between hydrodynamics and electromagnetism led (through the work of Thomson, Lorentz, Kramers, ... [10]) to the crucial distinction between the bare parameters, such as m_0, which enter the field theoretic equations, and the observed parameters, such as the inertial mass m.

A quantum field theory in $D = 4$ dimensions, is given by a classical action functional,

$$S\,(A) = \int \mathcal{L}\,(A)\, d^4x, \tag{4}$$

where A is a classical field and the Lagrangian is of the form,

$$\mathcal{L}\,(A) = (\partial A)^2/2 - \frac{m^2}{2}\, A^2 + \mathcal{L}_{\mathrm{int}}(A), \tag{5}$$

where $\mathcal{L}_{\mathrm{int}}(A)$ is usually a polynomial in A and possibly its derivatives. One way to describe the quantum fields $\phi(x)$, is by means of the time ordered Green's functions,

$$G_N(x_1, \ldots, x_N) = \langle\, 0\, | T\, \phi(x_1) \ldots \phi(x_N) |\, 0\, \rangle, \tag{6}$$

where the time ordering symbol T means that the $\phi(x_j)$'s are written in order of increasing time from right to left.

The probability amplitude of a classical field configuration A is given by,

$$e^{i\, \frac{S(A)}{\hbar}}, \tag{7}$$

and if one could ignore the renormalization problem, the Green's functions would be computed as

$$G_N(x_1, \ldots, x_N) = \mathcal{N} \int e^{i\, \frac{S(A)}{\hbar}}\; A(x_1) \ldots A(x_N)\, [dA], \tag{8}$$