Commun. Math. Phys. 223, 261 – 288 (2001)

Communications in
**Mathematical
Physics**

A Spin-Statistics Theorem for Quantum Fields on Curved Spacetime Manifolds in a Generally Covariant Framework

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Received: 1 March 2001 / Accepted: 28 May 2001

Abstract: A model-independent, locally generally covariant formulation of quantum field theory over four-dimensional, globally hyperbolic spacetimes will be given which generalizes similar, previous approaches. Here, a generally covariant quantum field theory is an assignment of quantum fields to globally hyperbolic spacetimes with spin-structure where each quantum field propagates on the spacetime to which it is assigned. Imposing very natural conditions such as local general covariance, existence of a causal dynamical law, fixed spinor- or tensor type for all quantum fields of the theory, and that the quantum field on Minkowski spacetime satisfies the usual conditions, it will be shown that a spin-statistics theorem holds: If for some of the spacetimes the corresponding quantum field obeys the "wrong" connection between spin and statistics, then all quantum fields of the theory, on each spacetime, are trivial.

1. Introduction

The spin-statistics theorem of quantum field theory in Minkowski spacetime asserts that elementary particles with integer spin must obey Bose-statistics ("spacelike commutativity"), while those of half-integer spin must obey Fermi-statistics ("spacelike anti-commutativity"). Although this behaviour of elementary particles is often taken as an experimental fact of life, it is remarkable that in quantum field theory such a connection between two at first sight apparently unrelated properties of particles can be deduced from a few very basic principles: (1) Relativistic covariance, (2) stability of matter (spectrum condition and existence of a vacuum state), (3) localization properties of charges and (4) locality (spacelike commutativity of observable quantities).

This deeply rooted connection between the covariance properties of elementary particles and the behaviour under exchange of their positions has attracted the attention of numerous researchers in quantum field theory, and has a long history with a fair number of general and rigorous results. Among the first are the investigations by Pauli [38] and by Fierz [20] who proved the spin-statistics theorem for quantum fields of

arbitrary spin obeying linear hyperbolic wave-equations in Minkowski-spacetime. The first results on the connection between spin and statistics in quantum field theory in a completely general, model-independent approach (for quantum fields in the Wightman framework) were then obtained by Burgoyne [11] and by Lüders and Zumino [36]. They have subsequently been further extended and refined, particularly to cover the situation of having several fields of different spinor types in a quantum field theory; these theorems are presented in the textbooks by Jost [33], by Streater and Wightman [44], and by Bogoliubov, Logunov, Todorov and Oksak [5], to which we refer the reader for further discussion and references.

The Wightman-framework takes as fundamental objects pointlike quantum fields which may be charge-carrying and need not represent observable quantities. The operator-algebraic approach to quantum field theory [30, 29] uses, instead, observable quantities as the basic objects describing a theory of elementary particles and, at the same time, abandons their pointlike localizability. The charge-carrying objects and the global gauge group are, in this approach, not put in by hand, but can be reconstructed from the observables together with sets of states distinguished by certain localization properties (representing the localization properties of the charges in a quantum field theory). This is a deep result by Doplicher and Roberts [16] arising from the profound analysis of the charge superselection structure by Doplicher, Haag and Roberts (see [15, 16, 29] and references given therein). Spin-statistics theorems have also been derived in the operator-algebraic approach to quantum field theory, beginning with works by Epstein [19] and by Doplicher, Haag and Roberts [15] for the case of strictly localizable charges. Generalizations of spin-statistics theorems to the case of charges that can be localized in spacelike cones have been obtained by Buchholz and Epstein [10].

A new line of development has been introduced by the Tomita–Takesaki modular theory of von Neumann algebras [46] and its connection to Lorentz-transformations which was first established in two articles by Bisognano and Wichmann [4]; see the recent review by Borchers [6] for more information on this nowadays very important area of activity in algebraic quantum field theory. In this context, there are spin-statistics theorems by Guido and Longo [26] and by Kuckert [35] in algebraic quantum field theory which take a certain geometric action of the Tomita–Takesaki modular objects associated with the vacuum state and distinguished algebras of quantum field observables as the starting point.

The results just summarized concern quantum field theory on four-dimensional Minkowski spacetime. The present article focusses on quantum field theory on four-dimensional curved spacetimes, but before turning to that topic, we just mention that spin-statistics connections have also been investigated in other settings. Among those are, in particular, quantum field theories on flat two-dimensional spacetime and chiral conformal quantum field theories on one-dimensional spacetimes (e.g. the circle S^1), see e.g. the articles [40] for the case of two dimensions and [27] for chiral conformal quantum field theory. A spin-statistics connection for so-called "topological geons" has been investigated within a diffeomorphism-covariant approach to quantum gravity [17, 2] which is not directly related to the quantum field theoretical framework. For the sake of completeness we mention that the spin-statistics connection may also be violated e.g. for quantum fields having infinitely many components; at this point we refer to [5] and references cited there.

While the spin-statistics connection is well-explored in quantum field theory on flat spacetime, offering a wealth of results, there is little analogous to be found so far for quantum field theory on curved spacetime manifolds. We recall that in quantum field