Stationarity of Lagrangian Velocity in Compressible Environments

Tomasz Komorowski

Institute of Mathematics, UMCS, Lublin, Poland

Received: 1 May 2001 / Accepted: 4 December 2001

Abstract: We study the transport of a passive tracer particle by a random $d$-dimensional, Gaussian, compressible velocity field. It is well known, since the work of Lumley, see [13], and Port and Stone, see [20], that the observations of the velocity field from the moving particle, the so-called Lagrangian velocity process, are statistically stationary when the field itself is incompressible. In this paper we study the question of stationarity of Lagrangian observations in compressible environments. We show that, given sufficient temporal decorrelation of the velocity statistics, there exists a transformation of the original probability measure, under which the Lagrangian velocity process is time stationary. The transformed probability is equivalent to the original measure. As an application of this result we prove the law of large numbers for the particle trajectory.

1. Introduction

Turbulent transport of a passive tracer is modeled by a stochastic differential equation with a random drift

$$
\begin{align*}
\frac{dX(t)}{dt} &= V(t, X(t)) + \sqrt{2\kappa} dw(t), \\
X(0) &= 0.
\end{align*}
$$

(1.1)

$V : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is assumed to be a $d$-dimensional, time-space stationary, random field over a certain probability space $\mathcal{T}_0 := (\Omega, \mathcal{V}, \mathbb{P})$ and $w(\cdot)$ is a standard $d$-dimensional Brownian motion, given over another probability space $\mathcal{T}_1 := (\Sigma, \mathcal{W}, \mathbb{W})$. The tracer particle trajectory $X(\cdot)$ is considered as a stochastic process over the probability space $\mathcal{T}_0 \otimes \mathcal{T}_1 := (\Omega \times \Sigma, \mathcal{V} \otimes \mathcal{W}, \mathbb{P} \otimes \mathbb{W})$. The parameter $\kappa \geq 0$, also called the molecular diffusivity, models the strength of the intrinsic diffusive dispersion of the medium.

Research supported by the State Committee for Scientific Research Grant Nr 2 PO3A 017 17.
The question of considerable interest in statistical hydrodynamics, see, e.g., [16, 14, 6], is the long-time, large scale behavior of the particle trajectory. An important tool in the investigation of this problem is the notion of an invariant measure for the Lagrangian velocity process

$$\eta_t := V(t, X(t)), \quad t \geq 0. \quad (1.2)$$

A measure $P_*$ is called to be invariant under the Lagrangian dynamics if the process $(\eta_t)_{t \geq 0}$ is stationary under $P_* \otimes W$.

As a matter of immediate observation note that if $P_*$ is invariant and $V(0, 0)$ has the first absolute moment w.r.t. $P_*$, then, thanks to the individual Ergodic Theorem, we can conclude the existence of the so-called Stokes drift

$$v := \lim_{t \uparrow +\infty} \frac{X(t)}{t}, \quad P_* \otimes W - a.s. \quad (1.3)$$

When, in addition, $P_*$ and $P$ are equivalent, i.e., $P[A] = 0$ iff $P_*[A] = 0$, the limit in (1.3) exists also $P \otimes W$-a.s. Furthermore, $v$ is deterministic if the Lagrangian process is $P_* \otimes W$-ergodic. An invariant probability measure $P_*$ that is both equivalent to the original probability and has the described ergodicity property shall be called regular.

Stationarity of the Lagrangian process has been established for incompressible velocities with $P_* = P$, see [13, 20]. This result has been crucial for proving homogenization in the case of incompressible flows, see, e.g., [18, 3, 5].

A regular invariant measure can also be found in some other special cases. For example, if a steady velocity field is the gradient of a stationary scalar potential, i.e., $V(x) = \nabla x \phi(x)$, where $\phi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a stationary field satisfying $Z := \int e^{-\phi(0;x)} dP < +\infty$, then the invariant measure can be found explicitly and it has a “Gibbs-like” form

$$P_*(d\omega) := \frac{1}{Z} e^{-\phi(0;\omega)} P(d\omega),$$

see, e.g., [17].

Another situation when the existence of an invariant measure can be established is the case of a field that is either periodic, or its temporal dynamics is driven by a finite dimensional diffusion. The latter means that $V(t, x) := W(\xi_t, x)$, where $W : \mathbb{R}^N \times \mathbb{R}^d \to \mathbb{R}^d$ is a deterministic vector field and $(\xi_t)_{t \geq 0}$ is an $N$-dimensional diffusion. In both of these cases the problem of finding an invariant measure can be resolved via finite dimensional techniques, see, e.g., [19].

In contrast with the special situations described above, proving the existence of a regular invariant measure for the Lagrangian velocity corresponding to a compressible field is very hard, due to an infinite dimensional character of the problem and, according to our knowledge, there are no general results in this direction in dimensions $d \geq 2$.

Recently, Sznitman and Zerner in [22], have proven the existence of an invariant measure for an “environment process”, i.e., the process describing the environment viewed from the moving particle, for a related problem of a nearest neighbor random walk in a steady (time independent) random environment. The random environment in question is a $d$-dimensional integer lattice with i.i.d. $2d$-dimensional random vectors – the transition probabilities – assigned to each site. The components of a random vector at a given site of the lattice represent the probabilities that the random walker jumps, in the next time step, to a respective neighboring site. These transition probabilities satisfy the uniform ellipticity condition, which corresponds to the case of $\kappa > 0$ considered here.