Dichotomy for the Hausdorff dimension of the set of nonergodic directions

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Abstract Given an irrational $0 < \lambda < 1$, we consider billiards in the table $P_\lambda$ formed by a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length $\frac{1-\lambda}{2}$ with one end touching at the midpoint of a vertical side. Let $\text{NE}(P_\lambda)$ be the set of $\theta$ such that the flow on $P_\lambda$ in direction $\theta$ is not ergodic. We show that the Hausdorff dimension of $\text{NE}(P_\lambda)$ can only take on the values $0$ and $\frac{1}{2}$, depending on the summability of the series $\sum_k \frac{\log \log q_{k+1}}{q_k}$ where $\{q_k\}$ is the sequence of denominators of the continued fraction expansion of $\lambda$. More specifically, we prove that the Hausdorff dimension is $\frac{1}{2}$ if this series converges, and $0$ otherwise. This extends earlier results of Boshernitzan and Cheung.

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1 Introduction

In 1969, Veech [17] found examples of skew products over a rotation of the circle that are minimal but not uniquely ergodic. These were turned into interval exchange transformations in [9]. Masur and Smillie gave a geometric interpretation of these examples (see for instance [14]) which may be described as follows. Let $P_\lambda$ denote the billiard in a $\frac{1}{2} \times 1$ rectangle with a horizontal barrier of length $\alpha = \frac{1-\lambda}{2}$ based at the midpoint of a vertical side. There is a standard unfolding procedure which turns billiards in this polygon into flows along parallel lines on a translation surface. See Fig. 1.

The associated translation surface in this case is a double cover of a standard flat torus of area one branched over two points $z_0$ and $z_1$ a horizontal distance $\lambda$ apart on the flat torus. See Fig. 2. We denote it by $(X, \omega)$.

The linear flows on this translation surface preserve Lebesgue measure. What Veech showed in these examples is that given $\theta$ with unbounded partial quotients in its continued fraction expansion, there is a $\lambda$ such that the flow on $P_\lambda$ in direction with slope $\theta$ is minimal but not uniquely ergodic.

Let $\text{NE}(P_\lambda)$ denote the set of nonergodic directions, i.e. those directions for which Lebesgue measure is not ergodic. It was shown in [14] that $\text{NE}(P_\lambda)$ is uncountable if $\lambda$ is irrational. When $\lambda$ is rational, a result of Veech [18] implies that minimal directions are uniquely ergodic; thus $\text{NE}(P_\lambda)$ is the set

![Unfolding the table $P_\lambda$](image.png)