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Stokes matrices, Poisson Lie groups and Frobenius manifolds

P.P. Boalch

S.I.S.S.A., Via Beirut 2–4, 34014 Trieste, Italy
(e-mail: boalch@sissa.it)

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1. Introduction

The purpose of this paper is to point out and then draw some consequences of the fact that the Poisson Lie group G^* dual to $G = GL_n(\mathbb{C})$ may be identified with a certain moduli space of meromorphic connections over the unit disc having an irregular singularity at the origin. (G^* will be fully described in Sect. 2.)

The key feature of this point of view is that there is a holomorphic map

$$\nu : \mathfrak{g}^* \longrightarrow G^*$$

from the dual of the Lie algebra to the group G^*, for each choice of diagonal matrix A_0 with distinct eigenvalues—the 'irregular type'. This map is essentially the Riemann-Hilbert map or de Rham morphism for such connections (we will call it the 'monodromy map'); it is generically a local analytic isomorphism. The main result is:

Theorem 1. *The monodromy map ν is a Poisson map for each choice of irregular type, where $\mathfrak{g}^*$ has its standard complex Poisson structure and G^* has its canonical complex Poisson Lie group structure, but scaled by a factor of $2\pi i$.*

This was conjectured, and proved in the simplest case, in [6] based on the observation that the space of monodromy/Stokes data of such irregular singular connections 'looks like' the group G^*, and that the symplectic leaves match up.

We will give two applications. First, although ν is neither injective or surjective, upon restricting to the skew-Hermitian matrices $\mathfrak{k}^* \subset \mathfrak{g}^*$ it becomes injective, at least when A_0 is purely imaginary, i.e. diagonal skew-Hermitian (both $\mathfrak{k}^*$ and $\mathfrak{g}^*$ are identified with their duals using the trace here). We also find that the involution $B \mapsto -B^\dagger$ fixing the skew-Hermitian

matrices corresponds under ν to an involution fixing the Poisson Lie group K^* dual to the unitary group $K = U(n)$. This leads to:

Theorem 2. *For each purely imaginary irregular type A_0 the monodromy map restricts to a (real) Poisson diffeomorphism $\mathfrak{k}^* \cong K^*$ from the dual of the Lie algebra of K to the dual Poisson Lie group (with its standard Poisson structure, scaled by a factor of π).*

Thus we have a new, direct proof of a theorem of Ginzburg and Weinstein [16], that $\mathfrak{k}^*$ and K^* are (globally) isomorphic as Poisson manifolds. Such diffeomorphisms enable one to convert Kostant's *non-linear* convexity theorem (involving the Iwasawa projection) into Kostant's *linear* convexity theorem (which is due to Schur and Horn in the unitary case, and led to the well-known Atiyah, Guillemin and Sternberg convexity theorem). See [21] and Sect. 6 below. Our approach also gives a new proof of a closely related theorem of Duistermaat [14], as well as a proof of a conjecture of Flaschka and Ratiu [15] concerning convexity theorems for non-Abelian group actions (see Remark 34 below).

Secondly (and this was our original motivation) if we restrict to skew-symmetric (complex) matrices then the corresponding space of Stokes data naturally appears as a moduli space of two-dimensional topological quantum field theories. This is due to B. Dubrovin: in [11] the notion of a *Frobenius manifold* is defined as a geometrical/coordinate-free manifestation of the WDVV equations of Witten-Dijkgraaf-Verlinde-Verlinde governing deformations of 2D topological field theories (see also [12, 13]). One of the main results (Theorem 3.2) of [11] is the identification of the local moduli of semisimple Frobenius manifolds with the entries of a Stokes matrix: an upper triangular matrix $S \in U_+$ with ones on the diagonal. An intriguing aspect of [11] was the explicit formula (F.21 in Appendix F) for a Poisson bracket on this space of matrices in the three dimensional case:

$$S := \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} \{x, y\} &= xy - 2z \\ \{y, z\} &= yz - 2x \\ \{z, x\} &= zx - 2y. \end{aligned} \tag{1}$$

This Poisson structure is invariant under a natural braid group action and has two-dimensional symplectic leaves parameterised by the values of the Markoff polynomial

$$x^2 + y^2 + z^2 - xyz.$$

For example, the quantum cohomology of the complex projective plane $\mathbb{P}^2(\mathbb{C})$ is a 3-dimensional semisimple Frobenius manifold and corresponds to the point $S = \left(\begin{smallmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. (The manifold is just the complex cohomology $H^*(\mathbb{P}^2)$ and the Frobenius structure comes from the 'quantum product', deforming the usual cup product.) This is an integer solution of the Markoff equation $x^2 + y^2 + z^2 - xyz = 0$ and quite surprisingly it follows that the solution of the WDVV equations corresponding to the quantum cohomology of $\mathbb{P}^2$ is not an algebraic function, from Markoff's proof (in