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Inventiones mathematicae

The non-commutative Weil algebra

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Abstract. For any compact Lie group G, together with an invariant inner product on its Lie algebra $\mathfrak{g}$, we define the non-commutative Weil algebra $\mathcal{W}_G$ as a tensor product of the universal enveloping algebra $U(\mathfrak{g})$ and the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. Just like the usual Weil algebra $W_G = S(\mathfrak{g}^*) \otimes \wedge\mathfrak{g}^*$, $\mathcal{W}_G$ carries the structure of an acyclic, locally free G-differential algebra and can be used to define equivariant cohomology $\mathcal{H}_G(B)$ for any G-differential algebra B. We construct an explicit isomorphism $\mathcal{Q} : W_G \to \mathcal{W}_G$ of the two Weil algebras as G-differential spaces, and prove that their multiplication maps are G-chain homotopic. This implies that the map in cohomology $H_G(B) \to \mathcal{H}_G(B)$ induced by $\mathcal{Q}$ is a ring isomorphism. For the trivial G-differential algebra $B = \mathbb{R}$, this reduces to the Duflo isomorphism $S(\mathfrak{g})^G \cong U(\mathfrak{g})^G$ between the ring of invariant polynomials and the ring of Casimir elements.

1. Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The Duflo map is a vector space isomorphism $\mathrm{Duf} : S(\mathfrak{g}) \to U(\mathfrak{g})$ between the symmetric algebra and the universal enveloping algebra which, as proved by Duflo [9], restricts to a *ring* isomorphism from the algebra of invariant polynomials $S(\mathfrak{g})^G$ onto the center $U(\mathfrak{g})^G$ of the universal enveloping algebra. For semi-simple $\mathfrak{g}$ the Duflo map coincides with the Harish-Chandra isomorphism. For generalizations of Duflo's theorem see the papers of Kashiwara-Vergne [15] and Kontsevich [16].

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In this paper we extend the Duflo map and Duflo's theorem to the equivariant cohomology of G-manifolds. Our result will involve a non-commutative version of the de Rham model of equivariant cohomology. Throughout we will assume that the group G is compact. Let

$$W_G = S\mathfrak{g}^* \otimes \wedge \mathfrak{g}^*$$

the Weil algebra of $\mathfrak{g}$. Given a basis e_a of $\mathfrak{g}$ let f_{ab}^c be the structure constants of $\mathfrak{g}$, and denote by v^a, y^a the generators of $S\mathfrak{g}^*$ and $\wedge\mathfrak{g}^*$ corresponding to the dual basis. The coadjoint action of G on $\mathfrak{g}^*$ induces an action on W_G, with generators L_a. Let derivations ι_a be given on generators by $\iota_a y^b = \delta_a^b$ and $\iota_a v^b = 0$. In [8], H. Cartan shows that the derivation

$$\mathrm{d}^W y^a = v^a - \frac{1}{2} f_{jk}^a y^j y^k, \quad \mathrm{d}^W v^a = - f_{jk}^a y^j v^k \tag{1}$$

gives W_G the structure of a G-differential algebra. In particular, d^W is a differential and $[\iota_a, \mathrm{d}^W] = L_a$.

Suppose $\mathfrak{g}$ comes equipped with a positive definite invariant inner product, used to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Let $\mathrm{Cl}(\mathfrak{g})$ be the corresponding Clifford algebra and let the *non-commutative Weil algebra* be its tensor product with the universal enveloping algebra:

$$\mathcal{W}_G = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

Take the basis e_a to be orthonormal, and let u_a, x_a be the corresponding generators of $U(\mathfrak{g})$ and $\mathrm{Cl}(\mathfrak{g})$. Again $\mathcal{W}_G$ carries a G-action induced by the adjoint action; let L_a be its generators and let ι_a be the derivation extension of $\iota_a x_b = \delta_{ab}$ and $\iota_a u_b = 0$. We show that there exists a derivation $\mathrm{d}^{\mathcal{W}}$ on $\mathcal{W}_G$ which on generators is given by formulas analogous to (1):

$$\mathrm{d}^{\mathcal{W}} x_a = u_a - \frac{1}{2} f_{ajk} x_j x_k, \quad \mathrm{d}^{\mathcal{W}} u_a = - f_{ajk} x_j u_k. \tag{2}$$

Moreover $\mathcal{W}_G$ is still a G-differential algebra, that is, $\mathrm{d}^{\mathcal{W}}$ squares to zero and Cartan's formula continues to hold. A new feature is that the derivations ι_a, L_a, d can be written as commutators. In particular, $\mathrm{d}^{\mathcal{W}} = \mathrm{ad}(\mathfrak{D})$ where

$$\mathfrak{D} = u_a x_a - \frac{1}{6} f_{abc} x_a x_b x_c$$

squares to the quadratic Casimir element

$$\mathfrak{D}^2 = \frac{1}{2} u_a u_a - \frac{1}{48} f_{abc} f_{abc}.$$

The latter is naturally interpreted as a Laplace operator on G and $\mathfrak{D}$ as a Dirac operator.