The contact boundary of a complex polynomial

Abstract. We define the contact boundary of a complex polynomial $f : \mathbb{C}^n \to \mathbb{C}$ as the intersection of some generic fiber with a large sphere. We show that, up to contact isotopy, this does not depend on the choice of the fiber (provided it is generic) and is invariant under polynomial automorphism of $\mathbb{C}^n$. We next prove that the formal homotopy class of this contact boundary is invariant in a large family of deformations of polynomials, which are not necessarily topologically trivial.

1. Introduction

In the local case, for a germ of a holomorphic function with an isolated singularity at the origin $g : (\mathbb{C}^n, 0) \to \mathbb{C}$, Milnor defined the link of $g$ as the intersection of $g^{-1}(0)$ with a small sphere centered at the origin [Mi]. It does not depend on the radius, provided this is small enough. Actually, the link is diffeomorphic to the slice by small spheres of the Milnor fiber (i.e. some nearby fiber $g^{-1}(\varepsilon)$). The topology of the Milnor fiber and of the link (together with its embedding into the sphere) yield important information about the singularity of $g$. Then Varchenko [V] showed that there is a natural contact structure on the link, which is invariant in analytic changes of coordinates. Recently, the first named author proved [C] that a certain contact invariant of the link, the formal homotopy class (see Example 2.7) is constant in deformations of $g$ with constant Milnor number (provided that $n \neq 3$).

In case of a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$, the study presents new difficulties: one cannot anymore concentrate only around the singularities of $f$ (supposing they are just isolated) since the topology of fibers may change also due to their asymptotic “bad” behaviour. In the last 15 years there has been significant progress in the understanding of the topology of polynomial functions. For instance, the topology of the link at infinity (called boundary in the sequel) of a “regular” fiber of $f$ has been extensively investigated, in case $n = 2$, by Eisenbud and Neumann [EN].
The new issue in our paper is the definition of the “generic” boundary of $f$ and its contact structure (Theorem 3.9, Definition 3.10), in all dimensions. We need to work with a certain class of strictly plurisubharmonic functions, out of which the squared distance function is an example. We show that not only a specific fiber with at most isolated singularities has a well defined contact boundary, but this structure is contact-isotopy invariant in the class of regular-at-infinity fibers (Definition 3.4). The genericity consists in the fact that the non-regular-at-infinity fibers are shown to be finitely many.

We next address the problem: to what extent the contact structure of the boundary preserves in suitable deformations. The class of deformations we consider (which we call V-deformations) verifies a natural condition: the typical fiber of the polynomial has the structure of a bouquet of $(n - 1)$-spheres. This is satisfied by polynomials of which all fibers are $\rho$-regular-at-infinity (see Definition 3.4 and [T2]) or polynomials which have isolated singularities at infinity in the sense of [ST]. We prove, using like in the local case the well-known result by Lê and Ramanaujam [LR], the invariance of the formal homotopy class of the contact boundary in such deformations, as long as the number of spheres in the bouquet decomposition remains constant (Theorem 4.3).

2. Relevant facts about contact and almost contact structures

In the following, all contact structures are defined by global 1-forms (i.e. they are coorientable).

2.1. Families of contact manifolds

We first recall the following result due to John W. Gray [G, Theorem 5.2.1], in a formulation which will be more suitable for our purposes (see e.g. [Ma]).

**Theorem 2.1 (J.W. Gray).** Let $\pi : M \to B$ be a smooth fiber bundle such that the fiber $M_b$ is a closed, oriented, odd-dimensional manifold. Suppose we are given a contact structure $\xi_b$ on each fiber $M_b$ which depends smoothly on the points $b$ of the base. Then the contact manifolds $(M_b, \xi_b)_{b \in B}$ are all contact isotopic.

Let us now give the two main applications of this theorem we will use in the sequel. We refer to [E1] for the definition and properties of plurisubharmonic functions on complex manifolds. Recall that the maximal complex hyperplane distribution on any smooth level set of a strictly plurisubharmonic function defines a contact structure (if $\varphi$ denotes the function and $J$ the complex structure, then a contact form is $\alpha := J^*d\varphi$).

**Proposition 2.2.** Let $W$ be a complex manifold and $\varphi : W \to \mathbb{R}$ a proper, strictly plurisubharmonic function. If $[a, b] \subset \mathbb{R}$ contains no critical value of $\varphi$, the levels $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$ are isotopic contact manifolds.

**Proof.** The function $\varphi$ being a proper submersion above $[a, b]$, it is a smooth fibration. Since any fiber is a contact manifold due to the strict plurisubharmonicity of $\varphi$, the conclusion follows directly from Theorem 2.1. $\square$