Indecomposable cycles on special quartics in $\mathbb{P}^3$

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Abstract. In this note we give an example of an indecomposable higher Chow cycle on a special family of quartics in $\mathbb{P}^3$. The example is obtained as an extension of a cycle in the higher Chow group $CH^2(K, 1)$ of a singular Kummer surface.

1. Introduction

Let $X$ be a quasi-projective variety over $\mathbb{C}$. S. Bloch has defined the higher Chow groups $CH_p(X, n)$, for $p, n \geq 0$, in [1]. There is a product map

$$CH^{p-m}(X, n-m) \otimes CH^1(X, 1) \otimes m \to CH^p(X, n).$$

The cycles in the group $\frac{CH_p(X, n)}{\text{Image}(\epsilon)}$ are called indecomposable cycles.

Existence of indecomposable cycles on a general quartic surface is known due to A. Collino and C. Oliva-C. Voisin ([3],[10]). There is some interest in finding indecomposable cycles in $CH^2(X, 1)$ for degenerations of $K3$-quartic surfaces ([8], p. 540–541) and on special quartics.

In this paper, we show that there is an indecomposable cycle on the singular Kummer’s quartic surface $K(C) \subset \mathbb{P}^3$, in $CH^2(K(C), 1)$, which extends in a family of non-singular quartics in $\mathbb{P}^3$.

More precisely: Let $C$ be a non-singular curve of genus 2 and $J(C)$ denote the Jacobian variety. The singular Kummer surface is the quotient of $J(C)$ by the inverse map and is naturally embedded as a quartic surface with 16-nodes in the linear system $|2C| = \mathbb{P}^3$. We observe that the cycle $Z$ defined in $CH^2(J(C), 1)$ by A. Collino ([2]) descends to the Kummer surface. There is an inclusion of quasi-projective varieties $K \subset S \subset \mathbb{P}^6$ where $K$ denotes the locus of Kummer surfaces and $S - K$ is the locus of non-singular invariant quartics $Q_s$ in $\mathbb{P}^3$ (here $\dim S = 6$ and $\dim K = 3$). The invariance is under a certain subgroup $H$ of the Heisenberg group $Heis(2, 2)$ acting on $\mathbb{P}^3$. In fact, consider a larger family $S \subset \tilde{S}$ of quartics such that the coordinate hyperplane sections are rational nodal curves and $\tilde{S} - K$ parameterises non-singular quartics. Consider the universal family $Q \to \tilde{S}$, $Q \subset \text{Proj}O_{\tilde{S}}[Y_0, Y_1, Y_2, Y_3]$.
Theorem 1.1. 1. There is a family of cycles $Z_s$ in $CH^2(Q_s, 1)\otimes \mathbb{Q}$, supported on $Q_s \cap \{Y_0Y_1 = 0\}$, for $s \in \tilde{S}$ such that $Z_s = W_s$, for $s \in K$. Here $W_s$ is Collino’s cycle on a Kummer surface.

2. For a general $s \in \tilde{S}$, the cycle $Z_s$ is indecomposable.

2. Higher Chow groups

2.1. Let $X$ be a quasi projective variety over $\mathbb{C}$ and $\Delta^n = \text{Spec} \mathbb{C}[\frac{X_0, \ldots, X_n}{1-\sum_j x_j}]$. Let $Z^p(X, n) \subset Z^p(X \times \Delta^n)$ be the subgroup containing algebraic cycles of codimension $p$ which intersects all faces $X \times \Delta^m, m < n$ in codimension at least $p$. Let $\delta_i : Z^p(X, n) \rightarrow Z^p(X, n - 1)$ be the restriction to the $i^{th}$ codimension 1 face (obtained by setting $t_i = 0$), for $0 \leq i \leq n$ and $\delta = \sum_i (-1)^i \delta_i$. The homology of the complex $[Z^p(X, \bullet), \delta]$ at $\bullet = n$ is the higher Chow group $CH^p(X, n)$ as defined by S. Bloch ([1]).

Suppose $X$ is a quasi-projective variety and $G$ is a finite group acting on $X$. Let $Y = \frac{X}{G}$ be the geometric quotient and $\pi : X \rightarrow Y$ be the quotient morphism. The action of $G$ can be extended to the product $X \times \Delta^n$, by letting $G$ act trivially on $\Delta^n$. Then $G$ acts on $Z^p(X \times \Delta^n)$ and also on $Z^p(X, n)$. We denote the subgroup of $G$-invariant cycles by $Z^p(X, n)^G$, for all $n \geq 0$.

Consider the complex (A):

$$\delta : Z^p(X, n + 1) \otimes \mathbb{Z}[\frac{1}{|G|}] \rightarrow Z^p(X, n) \otimes \mathbb{Z}[\frac{1}{|G|}]$$

$$\delta : Z^p(X, n - 1) \otimes \mathbb{Z}[\frac{1}{|G|}] \rightarrow$$

Applying the functor $-^G$ (associating the subgroup of $G$-invariants) to (A), we obtain the complex (B):

$$\delta : Z^p(X, n + 1)^G \otimes \mathbb{Z}[\frac{1}{|G|}] \rightarrow Z^p(X, n)^G \otimes \mathbb{Z}[\frac{1}{|G|}]$$

$$\delta : Z^p(X, n - 1)^G \otimes \mathbb{Z}[\frac{1}{|G|}] \rightarrow$$

Denote the homology by $CH^p(X, n)^G$.

Proposition 2.1. There is an isomorphism $CH^p(Y, n) \otimes \mathbb{Z}[\frac{1}{|G|}] \rightarrow CH^p(X, n)^G$.

Proof. The proof is similar to [5], p. 20–21, given for $n = 0$. Let $W \in Z^p(X, n)$ be an irreducible cycle and $G_W = \{g \in G : g(W) = W\}$ be the stabilizer. We denote the maps $X \times \Delta^n \xrightarrow{\pi \times Id} Y \times \Delta^n$ by $\pi$. For an irreducible cycle $V \in Z^p(Y, n)$, set $\pi^*(V) = \sum |G_W| \cdot W$, where the sum is over all the irreducible components $W$ of $\pi^{-1}V$. This gives a map $\pi^* : Z^p(Y, n) \rightarrow Z^p(X, n)^G$ which composed with the pushforward $\pi_* : Z^p(X, n) \rightarrow Z^p(Y, n)$, is multiplication by $\deg(\pi) = |G|$. 