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On the commutativity of $C^*$-algebras

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Abstract. Let $\mathcal{A}$ be a $C^*$-algebra. Let $f$ be a non-constant complex-valued continuous function defined on a closed interval $I$. We shall show that $f$ densely spans $\mathcal{A}$. As an application, $\mathcal{A}$ is commutative if $f(x)f(y) = f(y)f(x)$ for all self-adjoint elements $x$ and $y$ in $\mathcal{A}$ with spectrums contained in $I$.

For a $C^*$-algebra $\mathcal{A}$, the commutativity of $\mathcal{A}$ is simply the validity of the polynomial identity $xy = yx$ for all $x, y$ in $\mathcal{A}$. The theorem of Kaplansky [2, p. 68] states that $\mathcal{A}$ is commutative if, and only if, 0 is the only nilpotent elements of $\mathcal{A}$. M. J. Crabb, J. Duncan and C. M. McGregor [1] give two numerical characterizations of commutativity by the result of Kaplansky. They also show that Kaplansky’s theorem reduced the proofs of those order characterizations given in [3, 6, 7] to simple computations. Y. Kato [4] and R. Nakamoto [5] used the same Kaplansky’s theorem to give the spectrum characterizations. Following the line of development, Wu [8] shows that $\mathcal{A}$ is abelian if, and only if, $\exp(x+y) = \exp(x)\exp(y)$ holds for all self-adjoint elements $x, y$ of $\mathcal{A}$.

In this short note, we show that $\mathcal{A}$ is abelian if, and only if, there are two non-constant continuous real functions $f, g$ such that $f(x)g(y) = g(y)f(x)$ holds for all self-adjoint elements $x, y$ in $\mathcal{A}$ for which $f(x)$ and $g(y)$ can be defined through functional calculus. Here $f$ and $g$ can be the same functions. This, in particular, extends the result of Wu [8] since the identity $\exp(x+y) = \exp(x)\exp(y)$ implies $\exp(x)\exp(y) = \exp(y)\exp(x)$. As an application, we also show that $\mathcal{A}$ is abelian if, and only if, for example, $\sin(x+y) = \sin x \cos y + \cos x \sin y$, or any trigonometric identity, holds for all self-adjoint elements $x, y$ in $\mathcal{A}$. Our proof differs from the one of Wu [8] who asserts that the operator monotonicity of the exponential function ensuring the commutativity of $\mathcal{A}$. Indeed, we make use of an elemen-
tary but seemingly new observation that every non-constant continuous function densely spans $\mathcal{A}$.

Let $\mathcal{A}$ be a $C^*$-algebra with an identity $1$. For an $a$ in $\mathcal{A}$, let $\sigma(a)$ denote the spectrum of $a$, and $C^*(a)$ the $C^*$-subalgebra generated by $a$ (and 1). Let $\mathcal{A}_{sa}$ denote the family of all self-adjoint elements of $\mathcal{A}$. If $X$ is compact, let $C(X)$ denote the commutative $C^*$-algebra of all continuous complex-valued functions defined on $X$. Note that if $a \in \mathcal{A}_{sa}$, then $C^*(a)$ is the (norm) closure of $\{p(a) : p(z) is a complex polynomial in $z\}$ and $C^*(a)$ is $\ast$-isomorphic to $C(\sigma(a))$.

**Definition 1.** Let $\mathcal{A}$ be a unital $C^*$-algebra and $f$ be a continuous complex-valued function on an interval $I$ of the real line $\mathbb{R}$. Let $f(\mathcal{A})$ denote the (complex linear) subspace spanned by $\{f(x) : x \in \mathcal{A}_{sa} and \sigma(x) \subseteq I\}$. We say that

1. $f$ densely spans $\mathcal{A}$ if $f(\mathcal{A})$ is dense in $\mathcal{A}$, i.e., $f(\mathcal{A}) = \mathcal{A}$,
2. $f$ totally spans $\mathcal{A}$ if $f(\mathcal{A}) = \mathcal{A}$.

**Theorem 2.** Let $\mathcal{A}$ be a $C^*$-algebra with identity and $f$ be a continuous complex-valued function on an interval $I$ of the real line $\mathbb{R}$. If $f$ is non-constant, then $f$ densely spans $\mathcal{A}$.

**Proof.** Without loss of generality, we can assume $I = [t_0, t_1]$ and $f(t_0) \neq f(t_1)$.

First, we discuss the special case: $\mathcal{A} = C(X)$ where $X$ is a compact subset of $\mathbb{R}$. Suppose that $f$ does not densely span $C(X)$, i.e., the (complex linear) subspace spanned by $\{f \circ g : g \in C(X) and \sigma(g) = g(X) \subseteq I\}$ is not dense in $C(X)$. Then there is a nonzero regular Borel measure $\mu$ on $X$ such that

$$\int_X f(g(x)) d\mu(x) = 0, \quad \forall g \in C(X) with \sigma(g) \subseteq I.$$ 

For any nonempty open interval $(\alpha, \beta)$, define $g : X \to \mathbb{R}$ by $g(t) = t_1$, $\forall t \in (\alpha, \beta) \cap X$, and $g(t) = t_0$ otherwise. Let $\{\alpha_n\}_n$ be a decreasing sequence and $\{\beta_n\}_n$ be an increasing one such that $\alpha < \alpha_n < \beta_n < \beta$, $\lim_{n \to \infty} \alpha_n = \alpha$ and $\lim_{n \to \infty} \beta_n = \beta$. Let $g_n : X \to [t_0, t_1]$ be a continuous function such that $g_n(t) = t_1$ on $[\alpha_n, \beta_n] \cap X$, $g_n(t) = t_0$ outside $(\alpha, \beta)$, and $t_0 \leq g_n \leq t_1$, where $1$ is the identity function in $C(X)$. By the Dominated Convergence Theorem, we have

$$0 = \lim_{n \to \infty} \int_X f(g_n(t)) d\mu(t) = \int_X f(g(t)) d\mu(t) = f(t_1)\mu((\alpha, \beta) \cap X) + f(t_0)\mu(X \setminus (\alpha, \beta)).$$ 

On the other hand,

$$0 = \int_X f(t_0 1(t)) d\mu(t) = f(t_0)\mu(X) = f(t_0)[\mu((\alpha, \beta) \cap X) + \mu(X \setminus (\alpha, \beta))].$$

Consequently, we have $(f(t_1) - f(t_0))\mu((\alpha, \beta) \cap X) = 0$ and thus $\mu((\alpha, \beta) \cap X) = 0$. Hence $\mu(O) = 0$ for all open subsets $O$ of $X$. Since $\mu$ is regular, $\mu \equiv 0$, a contraction. Consequently, $f(C(X))$ is dense in $C(X)$. 