Most real analytic Cauchy-Riemann manifolds are nonalgebraizable

Abstract. We give a simple argument to the effect that most germs of generic real analytic Cauchy-Riemann manifolds of positive CR dimension are not holomorphically embeddable into a generic real algebraic CR manifold of the same real codimension in a finite dimensional space. In particular, most such germs are not holomorphically equivalent to a germ of a generic real algebraic CR manifold.

Introduction

A smooth real submanifold $M \subset \mathbb{C}^n$ in a complex Euclidean space is said to be a generic Cauchy-Riemann (CR) submanifold of CR dimension $m$ and codimension $d$ ($m + d = n$) if it is locally near every point $x \in M$ defined by $d$ real equations $\rho_1 = 0, \ldots, \rho_d = 0$ satisfying $\partial \rho_1 \wedge \ldots \wedge \partial \rho_d \neq 0$. (Here $\partial \rho = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} dz_j$ and $\wedge = \wedge_{\mathbb{C}}$.) A germ $(M, x)$ is real analytic (respectively real algebraic) if it is defined locally near $x$ by real analytic (resp. real algebraic) functions. Germs $(M, x), (M', x')$ are holomorphically equivalent if there exists a biholomorphic map $f : U \to U'$ from a neighborhood $U$ of $x$ onto a neighborhood $U'$ of $x'$ with $f(x) = x'$ and $f(M \cap U) = M' \cap U'$.

Beginning with Ebenfelt [4] (1996) several authors have given examples of analytic CR manifolds which are not locally holomorphically equivalent to an algebraic one (Baouendi, Ebenfelt and Rothschild ([1], 9.11.4), ([2], 7.2); Huang, Ji and Yau [8]). S. Ji studied the algebraization problem for real analytic strongly pseudoconvex hypersurfaces and established the propagation of algebraization for hypersurfaces with maximal Cartan-Chern-Moser rank [9], [10]. Gaussier and Merker studied the problem for a class of tuboids [6].

While it seems rather difficult to decide whether a specific analytic CR manifold is locally holomorphically equivalent to an algebraic one, the phenomenon itself is not at all surprising. The purpose of this note is to give a very simple argument to the effect that most germs of generic real analytic CR manifolds $M \subset \mathbb{C}^n$ of positive CR dimension are not holomorphically embeddable into any generic real algebraic CR manifold $M' \subset \mathbb{C}'$ of the same codimension as $M$; in particular,
they are not holomorphically equivalent to a germ of a real algebraic CR manifold in $\mathbb{C}^n$. More precisely, the embeddable ones form a set of the first category in a suitable Baire space (theorem 1.2). The same conclusion holds for embeddings into any countable union of finite dimensional families of CR manifolds of the same codimension.

Our proof employs an argument from [5] (which essentially goes back to Poincaré [12]) where it was proved that most germs of real analytic strongly pseudo-convex hypersurfaces in $\mathbb{C}^n$ for $n > 1$ are not holomorphically embeddable into any sphere $\sum_{j=1}^N |z_j|^2 = 1$ (Theorem 2.2 in [5]). For embeddings into infinite dimensional spheres see Lempert [11].

1. The main result

Every germ of analytic CR manifold in $\mathbb{C}^n$ of CR dimension $m$ and codimension $d$, with $m + d = n$, is holomorphically equivalent to one of the form

$$M = \{v_j = r_j(x, y, u) : j = 1, \ldots, d\}$$

where $z = x + iy \in \mathbb{C}^m$, $w = u + iv \in \mathbb{C}^d$ and $r = (r_1, \ldots, r_d)$ is an $\mathbb{R}^d$-valued convergent power series without constant and linear terms. Let $\mathcal{R}$ denote the space of all formal power series

$$r(x, y, u) = \sum_{\alpha, \beta, \gamma} c_{\alpha, \beta, \gamma} x^\alpha y^\beta u^\gamma (c_{\alpha, \beta, \gamma} \in \mathbb{R}^d)$$

without constant and linear terms in $2m + d$ real variables $(x, y, u)$. (One could put $M$ in a Chern-Moser normal form [3], although this will not be necessary for our purposes.) We shall identify $r \in \mathcal{R}$ with the (formal) germ at $0 \in \mathbb{C}^n$ of the CR manifold (1). $\mathcal{R}$ is a Fréchet space in the topology induced by the seminorms $||r||_{\alpha, \beta, \gamma} = |c_{\alpha, \beta, \gamma}|$ for all multiindices $\alpha, \beta \in \mathbb{Z}_+^m$, $\gamma \in \mathbb{Z}_+^d$. The convergent power series, representing germs of real analytic CR manifolds, form a union $\bigcup_{t>0} \mathcal{R}_t \subset \mathcal{R}$ of Banach spaces (in fact, Banach algebras)

$$\mathcal{R}_t = \{r \in \mathcal{R} : ||r||_t = \sum_{\alpha, \beta, \gamma} |c_{\alpha, \beta, \gamma}| \cdot t^{\alpha + |\beta| + |\gamma|} < +\infty\}$$

with the norm $||r||_t$ ([7], p. 15). For $r \in \mathcal{R}$ and $k \in \mathbb{N}$ we denote by $r_k$ its the truncation (Taylor polynomial) of order $k$. Let $\mathcal{R}_k$ be the (finite dimensional real) vector space of all such truncations.

Definition 1.1. A manifold $M$ (1) is embeddable into an algebraic model if there exists a real algebraic CR manifold $M' \subset \mathbb{C}^{n'} (n' \geq n)$ of real codimension $d$ and a holomorphic embedding $F = (f_1, \ldots, f_n): U \to \mathbb{C}^{n'}$, defined in an open neighborhood $U \subset \mathbb{C}^n$ of 0, such that $F$ is transverse to $M'$ at 0 and $F(M \cap U) = M' \cap F(U)$.

One defines formal holomorphic embeddability of a jet (1) into a similar jet $M' \subset \mathbb{C}^{n'}$ defined by $v' = \rho(x', y', u')$ by requiring that the composition $\rho \circ F$ is formally holomorphically equivalent to the jet (1) (see [5]).