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Stringy $E$-functions of varieties with $A$-$D$-$E$ singularities

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Abstract. The stringy $E$-function for normal irreducible complex varieties with at worst log terminal singularities was introduced by Batyrev. It is defined by data from a log resolution. If the variety is projective and Gorenstein and the stringy $E$-function is a polynomial, Batyrev also defined the stringy Hodge numbers as a generalization of the Hodge numbers of nonsingular projective varieties, and conjectured that they are nonnegative. We compute explicit formulae for the contribution of an $A$-$D$-$E$ singularity to the stringy $E$-function in arbitrary dimension. With these results we can say when the stringy $E$-function of a variety with such singularities is a polynomial and in that case we prove that the stringy Hodge numbers are nonnegative.

1. Introduction

1.1.

In [Ba1], Batyrev defined the stringy $E$-function for normal irreducible complex algebraic varieties, with at worst log terminal singularities. With this function he was able to formulate a topological mirror symmetry test for Calabi-Yau varieties with singularities. Before stating the definition of the stringy $E$-function, we recall some other definitions.

Let $X$ be a complex algebraic variety. One defines the Hodge-Deligne polynomial $H(X; u, v) \in \mathbb{Z}[u, v]$ by

$$H(X; u, v) = \sum_{i=0}^{2d} (-1)^i \sum_{p+q=i} h^{p,q}(H^i_c(X, \mathbb{C})) u^p v^q,$$

where $h^{p,q}$ denotes the dimension of the $(p, q)$-component of the mixed Hodge structure on $H^i_c(X, \mathbb{C})$. A nice introduction to Deligne’s mixed Hodge theory and to this definition can be found in [Sr] (pay attention to the extra factor $(-1)^{p+q}$ that the author has inserted there). The Hodge-Deligne polynomial is a generalized Euler characteristic, that is, it satisfies:

(1) $H(X) = H(X \setminus Y) + H(Y)$ where $Y$ is Zariski-closed in $X$,

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(2) $H(X \times X') = H(X) \cdot H(X')$.

Note that $H(X; 1, 1) = \chi(X)$, the topological Euler characteristic of $X$.

1.2. A normal irreducible complex variety $X$ is called $\mathbb{Q}$-Gorenstein if $rK_X$ is Cartier for some $r \in \mathbb{Z}_{>0}$. Take a log resolution $\phi: \tilde{X} \to X$ (i.e. a proper birational morphism from a nonsingular variety $\tilde{X}$ such that the exceptional locus of $\phi$ is a divisor whose components $D_1, \ldots, D_s$ are smooth and have normal crossings). Then we have $rK_{\tilde{X}} - \phi^*(rK_X) = \sum b_i D_i$, with $b_i \in \mathbb{Z}$. This is also formally written as $K_{\tilde{X}} - \phi^*(K_X) = \sum a_i D_i$, where $a_i = \frac{b_i}{r}$. The variety $X$ is called terminal, canonical, log terminal and log canonical if $a_i > 0$, $a_i \geq 0$, $a_i > -1$, $a_i \geq -1$, respectively, for all $i$ (this is independent of the chosen log resolution). The difference $K_{\tilde{X}} - \phi^*(K_X)$ is called the discrepancy.

1.3. Now we are ready to define the stringy $E$-function. We discuss its properties and give the additional definitions of the stringy Euler number and the stringy Hodge numbers. All of this goes back to Batyrev [Ba1].

Definition 1. Let $X$ be a normal irreducible complex variety with at most log terminal singularities and let $\phi: \tilde{X} \to X$ be a log resolution. Denote the irreducible components of the exceptional locus by $D_i$, $i \in I$, and write $D_J$ for $\cap_{j \in J} D_j$ and $D^J$ for $D_j \setminus \cup_{j \in I \setminus J} D_j$, where $J$ is any subset of $I$ ($D_\emptyset$ is taken to be $\tilde{X}$). The stringy $E$-function of $X$ is

$$E_{st}(X; u, v) := \sum_{J \subseteq I} H(D^J; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j} - 1},$$

where $a_j$ is the discrepancy coefficient of $D_j$ and where the product $\prod_{j \in J}$ is 1 if $J = \emptyset$.

Batyrev proved that this definition is independent of the chosen log resolution. His proof uses motivic integration. An overview of this theory is provided in [Ve1].

Remark.

(1) If $X$ is smooth, then $E_{st}(X) = H(X)$ and if $X$ admits a crepant resolution $\phi: \tilde{X} \to X$ (i.e. such that the discrepancy is 0), then $E_{st}(X) = H(\tilde{X})$.

(2) If $X$ is Gorenstein (i.e. $K_X$ is Cartier), then all $a_i \in \mathbb{Z}_{>0}$ and $E_{st}(X)$ becomes a rational function in $u$ and $v$. It is then an element of $\mathbb{Z}[[u, v]] \cap \mathbb{Q}(u, v)$.

(3) The stringy Euler number of $X$ is defined as

$$\lim_{u, v \to 1} E_{st}(X; u, v) = \sum_{J \subseteq I} \chi(D^J) \prod_{j \in J} \frac{1}{a_j + 1},$$