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Remarks on a paper of M. Ochiai

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Abstract. This note is related to a nice short paper of M. Ochiai. We prove in a very fast way that the two-parameter family of Heegaard diagrams, constructed by Ochiai, encodes the genuine three-sphere. The result is obtained, up to isotopy, by using a sequence of only three moves in this order: a Whitehead–Zieschang reduction, a band sum and a cancellation of a handle.

1. Introduction

In the short nice paper [5], Ochiai constructed a Heegaard diagram $\psi = (F; v, w)$ of the three-sphere which induces the following balanced presentation of the trivial group:

$$H = \langle X, Y, Z : XZY^2Z^{-1}X^{-1}(Y^{-1}Z^{-1})^3 = 1, \quad \text{with} \quad ZY(XZ)^4Y^{-1}Z^{-1}X^{-3} = 1, \quad XZY^3Z^{-1}X^{-1}(Y^{-1}Z^{-1})^4 = 1 \rangle.$$

The diagram $\psi$ has neither waves nor pairs of complementary handles. So it is not directly reducible by a wave move or by a Singer move of type III’ (see [4, 6]). Moreover, $\psi$ is a counterexample to the Whitehead conjecture [8] which asserts that all Heegaard diagrams of $S^3$, other than the canonical one, have always waves; this is proved to be true for Heegaard diagrams of genus two (see [4]). The counterexample also disproves the algorithm $A$ of [7] for recognizing $S^3$ among closed three-manifolds (the algorithm works in the case of genus two, as proved again in Homma et al. [4]). The diagram $\psi$ was related in Bandieri and Predieri [1] to special colored graphs, called crystallizations. The authors constructed a crystallization $(\Gamma, \gamma)$ of $S^3$ having $H$ as an associated presentation of the fundamental group with respect to a suitable choice of generators and relators. Moreover, $\psi$ is one among the Heegaard diagrams associated with $\Gamma$ (we refer also to Cavicchioli [2] for relations among Heegaard diagrams, spines and crystallizations). Then they proved that at least one among the remaining Heegaard diagrams, associated with $\Gamma$, has some pairs of complementary handles. This permits to reduce it by Singer

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moves of type III’ to the canonical diagram of $S^3$. However, the construction was based on a long procedure which also uses a computer program for simplifying crystallizations (see [1], p. 440).

As remarked in Ochiai [5], pp. 872–873, the diagram $\psi$ does not permit us to reduce directly the Heegaard genus of it, but permits us to construct a two-parameter family of Heegaard diagrams $\psi(n, m)$ having arbitrary many intersections of one complete system of meridians and another one. Such Heegaard diagrams induce the following balanced presentations of the trivial fundamental group (see [5], p. 873):

$$H(n, m) = \langle X, Y, Z : XZ^nY^{-1}Z^{-1}X^{-1}(Y^{-1}Z^{-1})^n = 1, \quad ZY(XZ)^{m-1}Y^{-1}Z^{-1}X^{-m} = 1, \quad XZ^nY^{-1}Z^{-1}(Y^{-1}Z^{-1})^{n+1} = 1 \rangle.$$  

Of course, we have $\psi = \psi(3, 3)$ and $H = H(3, 3)$. The construction method of Ochiai [5] permits us to make homotopy three-spheres with complicated presentations of the fundamental group such that they have Heegaard diagrams of genus three, and might be homotopy three-spheres other than the three-sphere. This however is not the case of the diagrams $\psi(n, m)$. In fact, we will prove the following result:

**Theorem.** For any $n, m \geq 3$, let $\psi(n, m)$ be the two-parameter Heegaard diagrams of genus three, depicted in Fig. 1, which induce the Ochiai presentations $H(n, m)$ of the trivial fundamental group. Then $\psi(n, m)$ can be directly reduced, up to isotopy, to the canonical Heegaard diagram of genus two of the genuine three-sphere by a sequence of three moves in this order: a Whitehead–Zieschang reduction, a band sum (that is, a Singer move of type II), and a cancellation of a handle.

2. Proof of the theorem

We follow essentially notation and terminology of Ochiai [5]. Let $\psi(n, m)$ denote $(F; v, w)$. Then one complete system of meridians $v = v(n, m) = (v_1, v_2, v_3)$ is illustrated in Fig. 1. The Heegaard surface $F$ (which is a closed orientable surface of genus 3) is obtained from the diagram in Fig. 1 by identifying meridians $v_i$ with $\tilde{v}_i$, for every $i = 1, 2, 3$. The meridians are to be paired respecting the orientations, and so that the marked points in Fig. 1 match up. Let $C_i$ and $\tilde{C}_i$ denote the meridian disks whose boundaries are $v_i$ and $\tilde{v}_i$, respectively. At the same time, another complete system of meridians $w = w(n, m) = (w_1, w_2, w_3)$ arises from the arcs of the diagram which connect points lying on $C_i$ and/or $\tilde{C}_i$. Let $X, Y$ and $Z$ be the generators of the fundamental group which are represented by the meridians $v_1, v_2$ and $v_3$, respectively. Walking along the meridians $w_i$ coherently with their orientations yields exactly the relations of the balanced group presentation $H(n, m)$. Now we are going to modify the diagram $\psi(n, m)$ into another one $\varphi(n, m) = (F; a, b)$ of genus three which represents the same manifold. Let $a_2$ be the simple closed curve