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Codimension two singularities for representations of extended Dynkin quivers

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Abstract. Let $M$ and $N$ be two representations of an extended Dynkin quiver such that the orbit $O_N$ of $N$ is contained in the orbit closure $\overline{O}_M$ and has codimension two. We show that the pointed variety $(\overline{O}_M, N)$ is smoothly equivalent to a simple surface singularity of type $A_n$, or to the cone over a rational normal curve.

1. Introduction and the main results

Throughout the paper, $k$ denotes an algebraically closed field of arbitrary characteristic, and $Q = (Q_0, Q_1, s, e)$ is a finite quiver, i.e. $Q_0$ is a finite set of vertices and $Q_1$ is a finite set of arrows $\alpha : s(\alpha) \rightarrow e(\alpha)$, where $s(\alpha)$ and $e(\alpha)$ denote the starting and the ending vertex of $\alpha$, respectively. A representation $V$ of $Q$ over $k$ is a collection $(V(i); i \in Q_0)$ of finite dimensional $k$-vector spaces together with a collection $(V(\alpha) : V(s(\alpha)) \rightarrow V(e(\alpha)); \alpha \in Q_1)$ of $k$-linear maps. A morphism $f : V \rightarrow W$ between two representations is a collection $(f(i) : V(i) \rightarrow W(i); i \in Q_0)$ of $k$-linear maps such that

$$f(e(\alpha)) \circ V(\alpha) = W(\alpha) \circ f(s(\alpha))$$

for all $\alpha \in Q_1$.

The dimension vector of a representation $V$ of $Q$ is the vector

$$\text{dim } V = (\dim_k V(i)) \in \mathbb{N}^{Q_0}.$$

We denote the category of representations of $Q$ by $\text{rep}(Q)$, and for any vector $d = (d_i) \in \mathbb{N}^{Q_0}$

$$\text{rep}_Q(d) = \prod_{\alpha \in Q_1} M_{d_e(\alpha) \times d_s(\alpha)}(k)$$

is the vector space of representations $V$ of $Q$ with $V(i) = k^{d_i}, i \in Q_0$. The group

$$\text{GL}(d) = \prod_{i \in Q_0} \text{GL}_{d_i}(k)$$

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acts on $\text{rep}_Q(d)$ by

$$(g_i \star V)(\alpha) = g_{e(\alpha)} \cdot V(\alpha) \cdot g_{s(\alpha)}^{-1}.$$  

Given a representation $V$ of $Q$, we denote by $O_V$ the $\text{GL}(d)$-orbit in $\text{rep}_Q(d)$ consisting of the representations isomorphic to $V$, where $d = \dim V$. An interesting problem is to study singularities of the Zariski closure $\overline{O}_V$ of an orbit $O_V$ in $\text{rep}_Q(d)$.

Following Hesselink [7, Sect. 1.7] we call two pointed varieties $(X, x_0)$ and $(Y, y_0)$ smoothly equivalent if there are smooth morphisms $f : Z \to X$, $g : Z \to Y$ and a point $z_0 \in Z$ with $f(z_0) = x_0$ and $g(z_0) = y_0$. This is an equivalence relation and the equivalence classes will be denoted by $\text{Sing}(X, x_0)$ and called the types of singularities. Obviously the regular points of the varieties form one type of singularity, which we denote by $\text{Reg}$. Let $M$ and $N$ be representations in $\text{rep}_Q(d)$ such that $M$ degenerates to $N$ ($N$ is a degeneration of $M$), i.e. $O_N \subseteq \overline{O}_M$. We shall write $\text{Sing}(M, N)$ for $\text{Sing}(\overline{O}_M, n)$, where $n$ is an arbitrary point of $O_N$, and denote by $\text{codim}(M, N)$ the codimension of $O_N$ in $\overline{O}_M$. We refer to [1, 3, 13–17] for results in this direction. Some of the results are expressed in terms of finite dimensional modules over finitely generated associative $k$-algebras, so it needs an explanation: given a representation $V$ of $Q$, we associate a (left) module $\tilde{V}$ over the path algebra $kQ$ of $Q$, whose underlying vector space is $\bigoplus_{i \in Q_0} V(i)$. This leads to an equivalence between $\text{rep}(Q)$ and the category of finite dimensional $kQ$-modules. Moreover, the equivalence preserves degenerations (of representations and of modules, respectively) as well as their codimensions and types of singularities (see [2]). Applying [15, Theorem 1.1] (and the above geometric equivalence between representations of $Q$ and modules over $kQ$), we get $\text{Sing}(M, N) = \text{Reg}$ if $\text{codim}(M, N) = 1$.

We assume now that $\text{codim}(M, N) = 2$. It was shown recently ([16, Thm.1.3]) that $\text{Sing}(M, N) = \text{Reg}$ provided $Q$ is a Dynkin quiver. This leads to a natural question about $\text{Sing}(M, N)$ if $Q$ is an extended Dynkin quiver, i.e. one of the following quivers

\[\begin{align*}
\tilde{A}_n, n \geq 0 : & \quad \bullet \quad \cdots \quad \bullet \\
\tilde{D}_n, n \geq 4 : & \quad \bullet \quad \bullet \quad \cdots \quad \bullet \\
\tilde{E}_6 : & \quad \bullet \\
\tilde{E}_7 : & \quad \bullet \quad \cdots \quad \bullet \\
\tilde{E}_8 : & \quad \bullet \quad \cdots \quad \bullet
\end{align*}\]