Equivariant Riemann–Roch theorems for curves over perfect fields

Received: 8 April 2008 / Revised: 11 August 2008
Published online: 30 September 2008

Abstract. We prove an equivariant Riemann–Roch formula for divisors on algebraic curves over perfect fields. By reduction to the known case of curves over algebraically closed fields, we first show a preliminary formula with coefficients in \( \mathbb{Q} \). We then prove and shed some further light on a divisibility result that yields a formula with integral coefficients. Moreover, we give variants of the main theorem for equivariant locally free sheaves of higher rank.

0. Introduction

Let \( X \) be a smooth, projective, geometrically irreducible curve over a perfect field \( k \) and let \( G \) be a finite subgroup of the automorphism group \( \text{Aut}(X/k) \). For any locally free \( G \)-sheaf \( \mathcal{E} \) on \( X \), we are interested in computing the equivariant Euler characteristic

\[
\chi(G, X, \mathcal{E}) := [H^0(X, \mathcal{E})] - [H^1(X, \mathcal{E})] \in K_0(G, k),
\]

considered as an element of the Grothendieck group \( K_0(G, k) \) of finitely generated modules over the group ring \( k[G] \). The main example of a locally free \( G \)-sheaf we have in mind is the sheaf \( \mathcal{L}(D) \) associated with a \( G \)-equivariant divisor \( D = \sum_{P \in X} n_P P \) (that is \( n_{\sigma(P)} = n_P \) for all \( \sigma \in G \) and all \( P \in X \)). If two \( k[G] \)-modules are in the same class in \( K_0(G, k) \), they are not necessarily isomorphic when the characteristic of \( k \) divides the order of \( G \). In order to be able to determine the actual \( k[G] \)-isomorphism class of \( H^0(X, \mathcal{E}) \) or \( H^1(X, \mathcal{E}) \), we are therefore also interested in deriving conditions for \( \chi(G, X, \mathcal{E}) \) to lie in the Grothendieck group \( K_0(k[G]) \) of finitely generated \textit{projective} \( k[G] \)-modules and in computing \( \chi(G, X, \mathcal{E}) \) within \( K_0(k[G]) \).

The equivariant Riemann–Roch problem goes back to Chevalley and Weil [3], who described the \( G \)-structure of the space of global holomorphic differentials on a compact Riemann surface. Ellingsrud and Lønsted [4] found a formula for the equivariant Euler characteristic of an arbitrary \( G \)-sheaf on a curve over an algebraically closed field.
closed field of characteristic zero. Nakajima [11] and Kani [7] independently generalized this to curves over arbitrary algebraically closed fields, under the assumption that the canonical morphism \( X \to X/G \) be tamely ramified. These results have been revisited by Borne [1], who also found a formula that computes the difference between the equivariant Euler characteristics of two \( G \)-sheaves in the case of a wildly ramified cover \( X \to X/G \). In the same setting, formulae for the equivariant Euler characteristic of a single \( G \)-sheaf have been found by the second author ([8,9]). Using these formulae, new proofs for the results of Ellingsrud-Lønsted, Nakajima and Kani have been given [8].

In this paper, we concentrate on the case where the underlying field \( k \) is perfect. Our main theorem, Theorem 3.4, is an equivariant Riemann–Roch formula in \( K_0(k[G]) \) when the canonical morphism \( X \to X/G \) is weakly ramified and \( \mathcal{E} = \mathcal{L}(D) \) for some equivariant divisor \( D \). By reduction to the known case of curves over algebraically closed fields, we first show a preliminary formula with coefficients in \( \mathbb{Q} \). The divisibility result needed to obtain a formula with integral coefficients is then proved in two ways: Firstly, by applying the preliminary formula to suitably chosen equivariant divisors; and secondly, in two situations, by a local argument. The following paragraphs describe the content of each section in more detail.

It is well-known that a finitely generated \( k[G] \)-module \( M \) is projective if and only if \( M \otimes_k \bar{k} \) is a projective \( \bar{k}[G] \)-module. In Sect. 2 we give a variant of this fact for classes in \( K_0(G, k) \) rather than for \( k[G] \)-modules \( M \) (Corollary 7). This variant is much harder to prove and is an essential tool for the proof of our main result in Sect. 3.

The first results in Sect. 3 give both a sufficient condition and a necessary condition under which the equivariant Euler characteristic \( \chi(G, X, \mathcal{E}) \) lies in the image of the Cartan homomorphism \( c : K_0(G, k) \to K_0(k[G]) \). More precisely, when \( \mathcal{E} = \mathcal{L}(D) \) for an equivariant divisor \( D = \sum_{P \in X} n_P P \), this holds if the canonical projection \( \pi : X \to X/G \) is weakly ramified and \( n_P + 1 \) is divisible by the wild part \( e_P^w \) of the ramification index \( e_P \) for all \( P \in X \). When \( \pi \) is weakly ramified we furthermore derive from the corresponding result in [9] the existence of the so-called ramification module \( N_{G,X} \), a certain projective \( k[G] \)-module which embodies a global relation between the (local) representations \( m_P / m_P^2 \) of the inertia group \( I_P \) for \( P \in X \). If moreover \( D \) is an equivariant divisor as above, our main result, Theorem 12, expresses \( \chi(G, X, \mathcal{L}(D)) \) as an integral linear combination in \( K_0(k[G]) \) of the classes of \( N_{G,X} \), the regular representation \( k[G] \) and the projective \( k[G] \)-modules \( \text{Ind}_{G_P}^{G} (W_{P,d}) \) (for \( P \in X \) and \( d \geq 0 \)) where the projective \( k[G_P] \)-module \( W_{P,d} \) is defined by the following isomorphism of \( k[G_P] \)-modules:

\[
\text{Ind}_{I_P}^{G} (\text{Cov}((m_P / m_P^2)^{(-d)})) \cong \bigoplus W_{P,d};
\]

here Cov means taking the \( k[I_P] \)-projective cover and \( f_P \) denotes the residual degree.

Finding an equivariant Riemann–Roch formula without denominators amounts to showing that \( W_{P,d} \) exists, i.e. that the left-hand side of the above is “divisible by \( f_P \)”. To do this, we use our prototype formula \textit{with} denominators, formula (5), and