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**Existence of solutions to a higher dimensional mean-field equation on manifolds**

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**Abstract.** For $m \geq 1$ we prove an existence result for the equation

$$(-\Delta_g)^m u + \lambda = \lambda \frac{e^{2mu}}{\int_M e^{2mu} d\mu_g}$$

on a closed Riemannian manifold $(M, g)$ of dimension $2m$ for certain values of $\lambda$.

1. Introduction and statement of the main result

Let $T^2 \cong S^1 \times S^1$ be the two-dimensional flat torus of volume one. Motivated by the study of vortices in the Chern-Simons Gauge theory, Struwe and Tarantello [17] showed that for $\lambda \in ]4\pi, 2\pi^2[$, the following equation admits a non-trivial solution $^1$

$$-\Delta u + \lambda = \lambda \frac{e^{2u}}{\int_{T^2} e^{2u} dx} \quad \text{on } T^2. \quad (1)$$

In this paper we generalize this result by considering an arbitrary closed Riemannian manifold $(M, g)$ of dimension $2m$, and studying the equation

$$(-\Delta_g)^m u + \lambda = \lambda \frac{e^{2mu}}{\int_M e^{2mu} d\mu_g} \quad \text{on } M, \quad (2)$$

where $\Delta_g$ is the Laplace–Beltrami operator. The main theorem we shall prove is the following.

**Theorem 1.** Let $\lambda_1 = \lambda_1(M)$ be the smallest eigenvalue of $(-\Delta_g)^m$ and $\Lambda_1 := (2m - 1)! \operatorname{vol}(S^{2m})$. Assume that $\Lambda_1/\operatorname{vol}(M) < \lambda_1/(2m)$. Then for every $\lambda \in ]\Lambda_1/\operatorname{vol}(M), \lambda_1/(2m)[$, $\lambda \notin \frac{\Lambda_1 N}{\operatorname{vol}(M)}$, (2) has a non-constant solution.

$^1$ Actually [17] deals with the equation $-\Delta u + \lambda = \lambda \frac{e^{u}}{\int_{T^2} e^{u} dx}$, but upon defining $\tilde{u} := 2u$, $\tilde{\lambda} = 2\lambda$ one can pass from one equation to the other.

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It is easy to see that in the case when $M = T^{2m}$ is the flat torus of dimension $2m$, one has $\Lambda_1 / \text{vol}(M) < \lambda_1 / (2m)$ for every $m \geq 1$, hence the theorem applies.

Notice that given a solution $u$ to (2), $u + \alpha$ is also a solution for any constant $\alpha \in \mathbb{R}$, hence it is not restrictive to assume that $\int_M ud\mu_g = 0$. Moreover, by a simple scaling argument we can assume that $\text{vol}(M) = 1$.

Equation 2 is a model for the intensively studied problems of existence and compactness properties of elliptic equations of order 4 and higher with critical non-linearity. In fact, other than the result of Theorem 1 itself, also the proof is interesting, as it rests on some recent compactness results for equations arising in conformal geometry. For this reason we shall now briefly describe its strategy, which is inspired to [17].

Let us consider the space

$$E := \left\{ u \in H^m(M) : \int_M ud\mu_g = 0 \right\},$$

with the norm

$$\|u\| := \left( \int_M |\Delta_g^m u|^2 d\mu_g \right)^{\frac{1}{2}},$$

where $\Delta_g^k u := \nabla_g \Delta_g^{k-1} u$ if $k$ is odd. Then weak solutions of (2) are critical points of the functional

$$I_\lambda(u) = \frac{1}{2} \int_M |\Delta_g^m u|^2 d\mu_g - \frac{\lambda}{2m} \log \left( \int_M e^{2mu} d\mu_g \right)$$

on $E$. By the Adams–Moser–Trudinger inequality (see [1] and Fontana [8]), we have

$$\sup_{u \in E} \int_M e^{m\Lambda_1 \|u\|^2} d\mu_g < \infty,$$

where $\Lambda_1 = (2m - 1)! \text{vol}(S^{2m})$ is the total $Q$-curvature of the round sphere of dimension $2m$, see e.g. [12]. Then writing $2mu \leq m \Lambda_1 \frac{\|u\|^2}{\|u\|^2} + \frac{m}{\Lambda_1} \|u\|^2$, we find

$$I_\lambda(u) \geq \left( \frac{1}{2} - \frac{\lambda}{2\Lambda_1} \right) \|u\|^2 - C.$$

Therefore $I_\lambda$ is bounded from below and coercive on $E$ for $\lambda \leq \Lambda_1$.

We shall see (Lemma 1) that $u \equiv 0$, which is a trivial solution to (2), is a strict local minimum of $I_\lambda$ if $\lambda < \lambda_1 / 2m$. Moreover for $\lambda > \Lambda_1$ there always exists a function $u \in E$ such that $I_\lambda(u) < I_\lambda(0) = 0$ (Lemma 2). This suggests that a mountain-pass technique might be used. In fact, as in [17], one can use a technique