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Evaluating Azumaya algebras on cubic surfaces

Received: 26 March 2010 / Revised: 27 July 2010
Published online: 10 October 2010

Abstract. Let $X$ be a cubic surface over a $p$-adic field $k$. Given an Azumaya algebra on $X$, we describe the local evaluation map $X(k) \to \mathbb{Q}/\mathbb{Z}$ in two cases, showing a sharp dependence on the geometry of the reduction of $X$. When $X$ has good reduction, then the evaluation map is constant. When the reduction of $X$ is a cone over a smooth cubic curve, then generically the evaluation map takes as many values as possible. We show that such a cubic surface defined over a number field has no Brauer–Manin obstruction. This extends results of Colliot-Thélène, Kanevsky and Sansuc.

1. Introduction

Let $L$ be a number field, and let $X \subset \mathbb{P}^3_L$ be a smooth cubic surface defined over $L$. It is known that $X$ does not have to satisfy the Hasse principle: that is, it is possible for $X(L_v)$ to be non-empty for each place $v$ of $L$, but for $X$ nonetheless to have no $L$-rational points; the first example of such a surface was given by Swinnerton-Dyer [17]. This and other counterexamples to the Hasse principle were shown by Manin [14] to be explained by what is now known as the Brauer–Manin obstruction. It has been conjectured by Colliot-Thélène that the Brauer–Manin obstruction is in fact the only obstruction to the Hasse principle for cubic surfaces (and, more generally, for geometrically rational varieties: see [5, p. 319]).

Let $\mathcal{M}_L$ denote the set of places of $L$. Let $\text{Br} X$ denote the Brauer group of $X$, and $X(\mathbb{A}_L)$ the set of adelic points; since $X$ is projective, $X(\mathbb{A}_L)$ is the same as $\prod_{v \in \mathcal{M}_L} X(L_v)$. The Brauer–Manin obstruction is based on the pairing

$$X(\mathbb{A}_L) \times \text{Br} X \to \mathbb{Q}/\mathbb{Z}, \quad ((x_v)_{v \in \mathcal{M}_L}, A) \mapsto \sum_{v \in \mathcal{M}_L} \text{inv}_v A(x_v). \quad (1)$$

Here $\text{inv}_v : \text{Br} L_v \to \mathbb{Q}/\mathbb{Z}$ is the invariant map of local class field theory: see [15, XIII, Sect. 3]. To understand the obstruction, then, it is desirable to have a good description of the local evaluation map $X(L_v) \to \mathbb{Q}/\mathbb{Z}, x \mapsto \text{inv}_v A(x)$, given by a particular element $A$ of the Brauer group of $X$ at some given place $v$. For a general variety, not much is known about this map, except that it is constant for $v$ outside
some finite set of primes; the usual way to compute it is simply to list the points of \(X(L_v)\) to sufficient accuracy and evaluate the invariant at each one. However, for curves the picture is much clearer: if \(C\) is a curve over a \(p\)-adic field \(k\), then the local pairing extends to a pairing \(\text{Pic} \times \text{Br} \to \text{Br} k\), which Lichtenbaum [12] showed can be identified with that arising from the Tate pairing, and is therefore non-degenerate.

In [6], Colliot-Thélène, Kanevsky and Sansuc gave a thorough description of both \(\text{Br} X\) and the local evaluation maps in the case when \(X\) is a diagonal cubic surface. In particular, they showed that the evaluation map is constant for places \(v\) where \(X\) has good reduction or where \(X\) is rational over \(L_v\); and, in contrast (Proposition 2), at finite places \(v\) where the reduction of \(X\) at \(v\) is a cone, the local evaluation map takes all possible values on \(X(L_v)\). In this last case the Brauer–Manin obstruction therefore vanishes. The proof relies on explicitly determining an Azumaya algebra which generates \(\text{Br} X/\text{Br} L\), showing that it has non-trivial restriction to a certain nonsingular cubic curve, and applying Lichtenbaum’s result.

The object of this article is to extend some of the results of [6] to more general cubic surfaces \(X\), and to show how these results can be deduced from the geometry of a model of \(X\). In particular, it is still straightforward to see that the local evaluation map is constant at places of good reduction: although this is already known, it gives a useful illustration of our approach; we prove this in Theorem 1. At places where \(X\) reduces to a cone, the behaviour described in [6] still happens as long as the singularity of the model of \(X\) is not too severe; this is proved in Theorem 2, which relies on a description of the geometry of the model given in Proposition 3.

1.1. Background

We now review the definition of the Brauer–Manin obstruction, partly in order to fix notation. An excellent reference for this topic is Skorobogatov’s book [16].

Define the Brauer group of a scheme \(X\) to be the étale cohomology group \(\text{Br} X = H^2(X, \mathbb{G}_m)\); for a smooth variety \(X\), this is equal to the group of equivalence classes of Azumaya algebras on \(X\). If \(K\) is any field, then a \(K\)-point of \(X\) corresponds to a morphism \(\text{Spec} K \to X\) and so by functoriality gives a homomorphism \(\text{Br} X \to \text{Br} K\). In this way we obtain an “evaluation” map \(X(K) \times \text{Br} X \to \text{Br} K\). In particular, suppose that \(X\) is a variety over a number field \(L\); then, for each place \(v\) of \(L\), there is a map \(X(L_v) \times \text{Br} X \to \text{Br} L_v\), which we may compose with the local invariant map \(\text{inv}_v : \text{Br} L_v \to \mathbb{Q}/\mathbb{Z}\). Let \(X(\mathbb{A}_L)\) denote the set of adelic points of \(X\), which is equal to the product \(\prod_v X(L_v)\) if \(X\) is projective. Adding together the local evaluation maps gives the map (1). Manin [13] observed that the \(L\)-rational points of \(X\) must lie in the left kernel of this map, and that this explained many known counterexamples to the Hasse principle—that is, varieties \(X\) with \(X(L_v) \neq \emptyset\) for all \(v\), yet \(X(L) = \emptyset\). This obstruction is known as the Brauer–Manin obstruction to the Hasse principle.

If \(X\) is any smooth, proper, geometrically integral variety over any field \(K\), there is an exact sequence as follows, which arises from the Hochschild–Serre spectral sequence for \(\mathbb{G}_m, X\):

\[
\text{Br} K \to \ker(\text{Br} X \to \text{Br} \bar{X}) \overset{\sim}{\to} H^1(K, \text{Pic} \bar{X}) \to H^3(K, \bar{K}^\times).
\] (2)