Constructing metrics on a 2-torus with a partially prescribed stable norm

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Abstract. A result of Bangert states that the stable norm associated to any Riemannian metric on the 2-torus $T^2$ is strictly convex. We demonstrate that the space of stable norms associated to metrics on $T^2$ forms a proper dense subset of the space of strictly convex norms on $\mathbb{R}^2$. In particular, given a strictly convex norm $\| \cdot \|_\infty$ on $\mathbb{R}^2$ we construct a sequence $\langle \| \cdot \|_j \rangle_1^\infty$ of stable norms that converge to $\| \cdot \|_\infty$ in the topology of compact convergence and have the property that for each $r > 0$ there is a $N = N(r)$ such that $\| \cdot \|_j$ agrees with $\| \cdot \|_\infty$ on $\mathbb{Z}^2 \cap \{(a, b) : a^2 + b^2 \leq r\}$ for all $j \geq N$. Using this result, we are able to derive results on multiplicities which arise in the minimum length spectrum of 2-tori and in the simple length spectrum of hyperbolic tori.

1. Introduction

Given a closed $n$-dimensional manifold $M$ with first Betti-number $b = b_1(M)$, we let $H_1(M; \mathbb{Z})_{\mathbb{R}}$ denote the collection of integral classes in the $b$-dimensional real vector space $H_1(M; \mathbb{R})$. Then $H_1(M; \mathbb{Z})_{\mathbb{R}}$ is a co-compact lattice in $H_1(M; \mathbb{R})$. Letting $T \simeq \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_q}$ denote the torsion subgroup of $H_1(M; \mathbb{Z}) \simeq \mathbb{Z}^b \times T$, we see that $H_1(M; \mathbb{Z})_{\mathbb{R}}$ can be identified with $H_1(M; \mathbb{Z})/T$ via the surjective homomorphism $\phi : H_1(M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})_{\mathbb{R}}$ given by

$$\sum_{i=1}^b z_i h_i + t \mapsto \left( \sum_{i=1}^b z_i h_i \right) \otimes \mathbb{Z} 1,$$

where $\{h_1, \ldots, h_b\}$ is some $\mathbb{Z}$-basis for $H_1(M; \mathbb{Z})$, the $z_i$’s are integers and $t \in T$. Now, let $\Psi : \pi_1(M) \rightarrow H_1(M; \mathbb{Z})$ denote the Hurewicz homomorphism [13], then the regular covering $p_{\text{Abel}} : M_{\text{Abel}} \rightarrow M$ of $M$ corresponding to $\ker(\Psi) = [\pi_1(M), \pi_1(M)]$ is the universal abelian covering of $M$. It is universal in the sense that it covers any other normal covering for which the deck transformations form
an abelian group. The universal torsion-free abelian cover \( p_{tor} : M_{tor} \to M \) corresponds to the normal subgroup \( \Psi^{-1}(T) < \pi_1(M) \): it covers all other normal coverings for which the group of deck transformations is torsion-free and abelian. Under the above identifications we see that the group of deck transformations of \( M_{tor} \to M \) is given by the lattice \( H_1(M; \mathbb{Z})_{\mathbb{R}} \). If \( M \) has positive first Betti number, then to each metric \( g \) we may associate a geometrically significant norm \( \| \cdot \|_s \) on \( H_1(M; \mathbb{R}) \) in the following manner.

For each \( h \in H_1(M; \mathbb{Z})_{\mathbb{R}} \cong \mathbb{Z}^b \leq H_1(M; \mathbb{R}) \) let
\[
f(h) = \inf \{ L_g(\sigma) : \sigma \text{ is a smooth loop representing the class } h \},
\]
where \( L_g \) is the length functional associated to the Riemannian metric \( g \) on \( M \).
Then, for each \( n \in \mathbb{N} \), we let \( f_n : \frac{1}{n} H_1(M; \mathbb{Z})_{\mathbb{R}} \to \mathbb{R}_+ \) be given by
\[
f_n(h) = \frac{1}{n} f(nh).
\]
It can be seen that the \( f_n \)'s converge uniformly on compact sets to a norm \( \| \cdot \|_s \) on \( H_1(M; \mathbb{R}) \) that is known as the stable norm of \( g \) [3]. In particular, if \( \{v_n\}_{n \in \mathbb{N}} \) is a sequence in \( H_1(M; \mathbb{Z})_{\mathbb{R}} \) such that \( \lim_{n \to \infty} \frac{v_n}{n} = v \in H_1(M; \mathbb{R}) \), then
\[
\|v\|_s = \lim_{n \to \infty} \frac{f(v_n)}{n}.
\]
An integral class \( v \in H_1(M; \mathbb{Z})_{\mathbb{R}} \) is said to be stable if there is an \( n \in \mathbb{N} \) such that \( \|v\|_s = f_n(v) = \frac{f(nv)}{n} \).

Intuitively, the stable norm \( \| \cdot \|_s \) describes the geometry of the universal torsion-free abelian cover \( (M_{tor}, g_{tor}) \) in a manner where the fundamental domain of the \( H_1(M; \mathbb{Z})_{\mathbb{R}} \)-action appears to be arbitrarily small. Indeed, for each \( n \in \mathbb{N} \), \( f_n \) is a (pseudo-)norm on the discrete group \( H_1(M; \mathbb{Z})_{\mathbb{R}} \) which illustrates the geometry of the fundamental domain of the \( H_1(M; \mathbb{Z})_{\mathbb{R}} \)-action on \( (M_{tor}, g_{tor}) \) when scaled by a factor of \( \frac{1}{n} \). And one can check that the sequence \( ((H_1(M; \mathbb{Z})_{\mathbb{R}}, f_n))_{n=1}^{\infty} \) of normed linear spaces converge to \( (H_1(M; \mathbb{R}), \| \cdot \|_s) \) in the Gromov-Hausdorff sense (cf. [11, p. 250]).

Now, let \( p : (N, h) \to (M, g) \) be a Riemannian covering. We will say that a non-constant geodesic \( \gamma : \mathbb{R} \to (M, g) \) is \( p \)-minimal (or minimal with respect to \( p \)) if for some and, hence, every lift \( \tilde{\gamma} : \mathbb{R} \to N \) of \( \gamma \), the geodesic \( \tilde{\gamma} \) is distance minimizing between any two of its points. That is, \( \gamma \) is \( p \)-minimal if for any \( t_1 \leq t_2 \) we have
\[
L_g(\tilde{\gamma}([t_1, t_2])) = d_N(\tilde{\gamma}(t_1), \tilde{\gamma}(t_2)).
\]
In the event that \( p \) is the universal Riemannian covering we will refer to \( p \)-minimal geodesics as minimal, and when \( \gamma \) is minimal with respect to the universal abelian cover \( p_{abel} : (M_{Abel}, h) \to (M, g) \) we will say that \( \gamma \) is an abelian minimal geodesic. In the case where \( \pi_1(M) \) is abelian—e.g., \( M \) is a torus—these two definitions coincide.

An interesting application of the stable norm \( \| \cdot \|_s \) is that characteristics of its unit ball \( B \subset H_1(T^2; \mathbb{Z}) \) can be used to deduce the existence (and properties) of minimal abelian geodesics. For instance, we have the following result due to Bangert.