YoungJu Choie, Kentaro Ihara

Iterated period integrals and multiple Hecke L-functions

Received: 23 June 2012 / Revised: 5 November 2012
Published online: 6 March 2013

Abstract. In this paper we express the multiple Hecke L-function in terms of a linear combination of iterated period integrals associated with elliptic cusp forms, which is introduced by Manin around 2004. This expression generalizes the classical formula of Hecke L-function obtained by the Mellin transformation of a cusp form. Also the expression gives a way of the analytic continuation of the multiple Hecke L-function.

1. Introduction

Recently, in [8,9], Manin introduces a generalization of the period integrals of elliptic cusp forms by means of the iterated path integrals on the complex upper-half plane and discusses several related topics. For example, he studies the iterated version of Mellin transformation and its functional equation, the properties of non-commutative generating series of the iterated integrals, an analogy of the classical period theory and its interpretation in terms of non-abelian group cohomology, and so on. The result which we would like to focus in his paper is an expression of the iterated period integral in terms of special values of a multiple Dirichlet series in their convergent region (see §3.2 in [8]).

In this paper first we define the multiple Hecke L-function, which is essentially the same as the Dirichlet series which he introduced, and show the analytic continuation of the function. Next, as a main result, we give the expression of the iterated period integral in terms of a linear combination of L-functions, which holds in arbitrary region, and generalizes Manin’s expression. This expression also generalizes the classical formula (1) below given by Mellin transformation. As a consequence, one can write the iterated period integral as a Q-linear combination of special values of L-function.

Let $\mathbf{H} = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \}$ be the complex upper-half plane. Assume that $f(z)$ is a holomorphic function on $\mathbf{H}$ satisfying following two conditions:

Y. Choie: Pohang Mathematics Institute, POSTECH Hyoja, San 31, 790-784 Pohang, Korea. e-mail: yjc@postech.ac.kr
K. Ihara (✉): Department of Mathematics, International College, Osaka University 1-30 Machikaneyama, Toyonaka, Osaka 560-0043, Japan. e-mail: k-ihara@math.sci.osaka-u.ac.jp

Mathematics Subject Classification (2010): Primary 11E45, Secondary 11M32

DOI: 10.1007/s00229-013-0605-2
(i) (Fourier expansion) \( f(z) \) is periodic with period one and has a Fourier expansion of the form \( f(z) = \sum_{m=1}^{\infty} c_m z^m \), where \( q = e^{2\pi i z} \) whose coefficients \( \{c_m\} \) have at most polynomial growth in \( m \) when \( m \to \infty \): \( c_m = O(m^M) \) for some \( M > 0 \).

(ii) (Cusp conditions) For any \( \gamma \in SL_2(\mathbb{Z}) \), there exists a constant \( C > 0 \) and an integer \( k \) such that

\[
(f|_k \gamma)(z) := j(\gamma, z)^{-k} f(\gamma z) = O(e^{2\pi i C \mathfrak{d}}), \quad z \to i\infty,
\]

where \( j(\gamma, z) = cz + d \) and \( \gamma z = (az + b)/(cz + d) \) for \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \).

For example, let \( \Gamma \) be a congruence subgroup of \( SL_2(\mathbb{Z}) \) containing the translation matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( S_k(\Gamma) \) be the space of holomorphic \( \Gamma \)-cusp forms of weight \( k \). Then \( f(z) \in S_k(\Gamma) \) satisfies both (i) and (ii).

The Mellin transformation of \( f(z) \) gives following expression:

\[
\int_{i\infty}^{0} f(z) z^{s-1} dz = -\Gamma(s)L(f, s) \tag{1}
\]

where \( \Gamma(s) = \int_{0}^{+\infty} e^{-t} t^{s-1} dt \) is the Gamma function and \( L(f, s) := (-2\pi i)^{-s} \sum_{m=1}^{\infty} c_m m^{-s} \) for \( \text{Re } s > 0 \) is the Hecke \( L \)-function attached to \( f \) which is normalized by \( (-2\pi i)^{-s} \) for the sake of convenience. (We take the non-positive imaginary axis as a branch cut in the \( z \)-plane, so that \( \arg(z) \in [-\pi/2, 3\pi/2] \) to define the complex power throughout this paper.) (Following [8,9], we chose \( i\infty \) as the base point of paths. The minus sign in RHS of (1) happens for this choice.) The condition (i) guarantees the convergence of the \( L \)-function in \( \text{Re } s > 0 \) and also (ii) implies the convergence of the integral in (1) in \( s \in \mathbb{C} \). For \( f \in S_k(\Gamma) \), the special values defined by the integral in (1) at critical points \( s = 1, \ldots, k-1 \) are called periods of \( f \) and play a fundamental role in the period theory. Shokurov [12] constructs some varieties, so called Kuga-Sato varieties, and interprets these numbers as periods of relative homology of the varieties. See also e.g. [6,7] for the period theory. The equation (1) gives not only the interpretation of the periods as the special values of \( L \)-function but the way of the analytic continuation of \( L \)-functions.

Manin generalizes the integral in (1) as follows. For \( r = 1, \ldots, n \), let \( f_r \) be holomorphic functions on \( \mathbb{H} \) satisfying the conditions (i) and (ii) above (for instance \( f_r \in S_k_r(\Gamma) \) for integers \( k_r \)) and \( s_r \) be complex variables. For points \( a, z \in \overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \), we fix a path joining \( a \) to \( z \) on \( \mathbb{H} \) and approaching \( a \) and \( z \) vertically if \( a \) or \( z \) is in \( \mathbb{P}^1(\mathbb{Q}) \). Then we consider the iterated integral along the path:

\[
I_a^z\left( f_1, \ldots, f_n \right) := \int_a^z f_1(z_1) z_1^{s_1-1} dz_1 \int_a^{z_1} f_2(z_2) z_2^{s_2-1} dz_2 \cdots \int_a^{z_{n-1}} f_n(z_n) z_n^{s_n-1} dz_n. \tag{2}
\]

The integral converges again because of cusp conditions of \( f_r \)'s and defines a holomorphic function in \( z \in \mathbb{H} \) and in \( (s_1, \ldots, s_n) \in \mathbb{C}^n \). For a connection between