Cohen–Macaulayness and squarefree modules

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Abstract. In this paper we consider the category of squarefree modules over the polynomial ring and an exact duality functor, which is an extension of the Alexander dual of a simplicial complex. We give a relationship between the squarefree components of local cohomology groups of a squarefree module and the Tor groups of its dual. With this result it is shown that a squarefree module is sequentially Cohen–Macaulay if and only if the dual is componentwise linear.

1. Introduction

Let \( \Delta \) be a simplicial complex on the vertex set \([n]\), \( K \) be a field, \( S = K[x_1, \ldots, x_n] \) be the polynomial ring and \( K[\Delta] \) be the Stanley–Reisner ring over \( S \). In a series of papers ([4, 6, 8, 9, 13]) relations between ring properties of \( K[\Delta] \) and those of the Stanley–Reisner ring \( K[\Delta^*] \) of the Alexander dual \( \Delta^* \) have been studied.

Recently Yanagawa [14] gave a definition of a squarefree \( S \)-module \( N \), which generalizes the concept of Stanley–Reisner rings. In [10] the author defined the generalized Alexander dual \( N^* \) for the squarefree modules. The definition refers to exterior algebras. In Section 1 we define more directly such a dual in the polynomial ring. It will be the kernel of a map which only depends on \( N \). Miller [7] studied Alexander duality in a more general situation. In the case of squarefree modules his definition and ours coincide.

This paper extends homological theorems on Stanley–Reisner rings. We show that a squarefree \( S \)-module \( N \) is Cohen–Macaulay of \( \dim N = d \), if and only if \( N^* \) has an \((n-d)\)-linear resolution and \( \text{proj dim}(N) = \text{reg}(N^*) \). This result generalizes theorems of Eagon–Reiner [6] and Terai [13]. Furthermore we generalize a result of Herzog–Hibi [9] and Herzog–Reiner–Welker [8] by showing that \( N \) is sequentially Cohen–Macaulay if and only if \( N^* \) is componentwise linear.

2. Squarefree modules

We fix some notation and recall some definitions. For \( a = (a_1, \ldots, a_n) \in \mathbb{N}^n \), we say \( a \) is squarefree if \( 0 \leq a_i \leq 1 \) for \( i = 1, \ldots, n \). We set \( |a| = a_1 + \ldots + a_n \).
and \( \text{supp}(a) = \{ i : a_i \neq 0 \} \subseteq [n] = \{ 1, \ldots, n \} \). Sometimes a squarefree vector \( a \) and \( F = \text{supp}(a) \) are identified. For \( F, G \subseteq [n] \) we set \( a(F, G) = \| (f, g) : f \in F, g \in G \| \). For an element \( u \) of an \( \mathbb{N}^n \)-graded vector space \( M = \bigoplus_{a \in \mathbb{N}^n} M_a \), the notation \( \deg(u) = a \) is equivalent to \( u \in M_a \); we set \( \text{supp}(\deg(a)) = \text{supp}(a) \) and \( |\deg(a)| = |a| \).

Let \( K \) be a field, \( S = K[x_1, \ldots, x_n] \) the symmetric algebra over \( K \) and \( m = (x_1, \ldots, x_n) \) the graded maximal ideal of \( S \). Consider the natural \( \mathbb{N}^n \)-grading on \( S \). For a monomial \( x_1^{a_1} \cdots x_n^{a_n} \) with \( a = (a_1, \ldots, a_n) \) we set \( x^a \).

Let \( E = K(e_1, \ldots, e_n) \) be the exterior algebra over an \( n \)-dimensional vector space \( V \) with basis \( e_1, \ldots, e_n \). We denote by \( \mathcal{M} \) the category of finitely generated graded left and right \( E \)-modules \( M \), satisfying \( ax = (-1)^{|\deg(a)|} \deg(x)x^a \) for all homogeneous \( a \in E \) and \( x \in M \). For example every graded ideal \( J \subseteq E \) belongs to \( \mathcal{M} \). For a module \( M \in \mathcal{M} \) we define \( M^* = \text{Hom}_E(M, E) \). Observe that \( \ast \) is an exact contravariant functor \( [3, 5.1 \text{ (a)}] \) and \( M^* \in \mathcal{M} \). For a \( K \)-vector space \( W \) we define \( W^\vee = \text{Hom}_K(W, K) \). The following was proved in \( [3, 5.1 \text{ (d)}] \): \((M^*)_\ast \cong (M_{\ast-1})^\vee \).

If \( a \in \mathbb{N}^n \) is squarefree we set \( e_a = e_{a_1} \wedge \cdots \wedge e_{a_n} \), where \( \text{supp}(a) = \{ j_1 < \ldots < j_l \} \) and we say \( e_a \) is a monomial in \( E \). For any \( a \in \mathbb{N}^n \) we set \( e_a = e_{\text{supp}(a)} \).

A simplicial complex \( \Delta \) is a collection of subsets of \( [n] \) such that \( \{ i \} \in \Delta \) for \( i = 1, \ldots, n \), and that \( F \in \Delta \) whenever \( F \subseteq G \) for some \( G \in \Delta \). Further we denote by \( \Delta^\ast = \{ F : F^\vee \not\in \Delta \} \) the Alexander dual of \( \Delta \), where \( F^\vee = [n] - F \). Then \( K[\Delta] = S/I_{\Delta} \) is the Stanley–Reisner ring, where \( I_{\Delta} = (x_{i_1} \cdots x_{i_s} : \{ i_1, \ldots, i_s \} \not\in \Delta) \), and \( K[\Delta] = E/J_{\Delta} \) is the exterior face ring, where \( J_{\Delta} = (e_{i_1} \wedge \cdots \wedge e_{i_s} : \{ i_1, \ldots, i_s \} \not\in \Delta) \).

The following definition is due to Yanagawa [14].

**Definition 2.1.** A finitely generated \( \mathbb{N}^n \)-graded \( S \)-module \( N = \bigoplus_{a \in \mathbb{N}^n} N_a \) is **squarefree** if the multiplication maps \( N_a \ni y \mapsto x_iy \in N_{a+e_i} \) is bijective for all \( a \in \mathbb{N}^n \) and all \( i \in \text{supp}(a) \).

For example the Stanley–Reisner ring \( K[\Delta] \) of a simplicial complex \( \Delta \) is a squarefree module. It is easy to see that for \( a \in \mathbb{N}^n \) and a squarefree module \( N \) we have \( \dim_K N_a = \dim_K N_{\text{supp}(a)} \) and \( N \) is generated by its squarefree part \( \{ N_a : a \subseteq [n] \} \).

Yanagawa proved in \([14, 2.3, 2.4] \) that if \( \varphi : N \to N' \) is a \( \mathbb{N}^n \)-homogeneous homomorphism, where \( N, N' \) are squarefree modules, \( \text{Ker}(\varphi) \) and \( \text{Coker}(\varphi) \) are again squarefree. It follows that every syzygy module \( \text{Syz}_2(N) \) in a multigraded minimal free \( S \)-resolution \( F_i \) of \( N \) is squarefree. Indeed the free \( S \)-module \( F_i \) is generated by homogeneous elements \( f \) such that \( \deg(f) \) is squarefree. Then \( F_i \) is called a squarefree resolution of \( N \). It follows that an \( S \)-module \( N \) is squarefree if and only if \( N \) has a squarefree resolution.

In the exterior algebra a squarefree module is defined as follows:

**Definition 2.2.** A finitely generated \( \mathbb{N}^n \)-graded \( E \)-module \( M = \bigoplus_{a \in \mathbb{N}^n} M_a \) is **squarefree** if it has only squarefree components.

For example the exterior face ring \( K[\Delta] \) obtained from a simplicial complex \( \Delta \) is a squarefree \( E \)-module.