The inverses of an H-space

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Abstract. A multiplication on an H-space $X$ has a left inverse $\lambda$ and a right inverse $\rho$. They are mutual inverses and $\lambda = \rho$ if and only if $\lambda^2 = id$. In this paper we investigate the order $|\lambda|$ of $\lambda$. We give an example of a multiplication with $|\lambda| = 6$, and prove that for any finite H-complex $X$ there are finitely many left inverses of finite order. Conditions are given for there to be infinitely many multiplications on $X$ with the same left inverse. We then give conditions for a left inverse to have infinite order. We apply these results to specific Lie groups.

1. Introduction

An $H$-space is a pair $(X, \mu)$ consisting of a based topological space $X$ and a homotopy class $\mu \in [X \times X, X]$, called the multiplication, whose restriction to each factor of $X \times X$ is $id$, the homotopy class of the identity map of $X$. If, in addition, $X$ is a (finite) CW-complex, we call $X$ or $(X, \mu)$ a (finite) $H$-complex. An H-space $(X, \mu)$ is group-like if $\mu$ is homotopy associative, i.e., $\mu(\mu \times id) = \mu(id \times \mu) \in [X \times X \times X, X]$ and if $\mu$ has a homotopy inverse. The latter condition is that there exists $\iota \in [X, X]$ such that $\mu(\iota \times id) \Delta = 0 = \mu(id \times \iota) \Delta$, where $\Delta$ is the homotopy class of the diagonal map of $X$ and $0$ is the constant homotopy class.

For any based space $A$, a multiplication $\mu$ of $X$ induces a binary operation on $[A, X]$ defined by $\alpha + \beta = \mu(\alpha \times \beta) \Delta$ such that $\alpha + 0 = 0 + \alpha = \alpha$. If $\mu$ is homotopy associative, then $[A, X]$ is associative. If $(X, \mu)$ is group-like, then $[A, X]$ is a group.

A great deal of work has been done on the homotopy associativity condition of an $H$-space [St1, St2, Za]. On the other hand, the inverse condition has been studied very little. The reasons for this may be the following: (1) the inverse condition rarely appears as a hypothesis in theorems about $H$-spaces (2) the result of James which we discuss next.

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Recall that an (algebraic) loop $L$ is a set with an additive binary operation such that for every $a, b \in L$ the equations $a + x = b$ and $y + a = b$ have unique solutions $x, y \in L$. Then James has proved [Ja3, Thm. 1.1] that $[A, X]$ is a loop if $A$ is a CW-complex and $(X, \mu)$ is an $H$-space. Thus every $\alpha \in [A, X]$ has a unique left inverse $\alpha_L$ and a unique right inverse $\alpha_R$ defined by $\alpha_L + \alpha = 0$ and $\alpha + \alpha_R = 0$. Hence if $(X, \mu)$ is an $H$-complex, there are unique elements $\lambda, \rho \in [X, X]$, called the left and right inverse of $\mu$, which are the left and right inverse of $id$, respectively, i.e., $\mu(\lambda \times id) \Delta = 0 = \mu(id \times \rho) \Delta$. It follows as in group theory that if $\mu$ is homotopy associative, then $\lambda = \rho$, and so $(X, \mu)$ is group-like. Thus the inverse property automatically holds for homotopy associative $H$-complexes, and this may be a reason why this condition has not received much attention. However, many familiar homotopy associative $H$-complexes such as compact, connected Lie groups, admit infinitely many multiplications which are not homotopy associative [Cu, Thm. II] and thus may have left and right inverses which are not equal [AL, Cor. 4.4]. It is therefore reasonable to study the inverse condition for arbitrary multiplications. We do this in this paper.

We next give a brief outline of the paper. We let $(X, \mu)$ be an $H$-complex with left inverse $\lambda$ and right inverse $\rho$. In §2 it is shown that $\lambda$ and $\rho$ are homotopy equivalences and mutual inverses and that $\lambda = \rho$ if and only if $\lambda \circ \lambda = \lambda^2 = id$. We construct a class of multiplications on any $H$-space all of which have the same left inverse and the same right inverse. In §3 we consider the order $|\lambda|$ of a left inverse $\lambda$ of a multiplication on $X$, i.e., the smallest positive integer $n$ such that $\lambda^n = id$; or $\infty$, if there is no such integer. Since $|\lambda| = 2$ corresponds to $\lambda = \rho$, we regard the order of $\lambda$ as a measure of how much $\lambda$ differs from $\rho$. Since $\lambda = \rho$ for homotopy associative multiplications, we could also regard a large order $|\lambda|$ as indicating that the multiplication is highly nonhomotopy associative. In §4 we use methods of rational homotopy theory to study inverses on an $H$-space $X$ by investigating multiplications and inverses on the Sullivan minimal model $M$ of $X$. We prove that for any finite $H$-complex, the set of all left inverses of finite order is a finite set. We obtain an easily verifiable condition on a homotopy associative finite $H$-complex $X$ (in terms of the rational cohomology of $X$) for there to be infinitely many multiplications with the same left inverse. This leads to a determination of which 1–connected simple Lie groups have this property. Finally, we give necessary and sufficient conditions for $X$ to admit a multiplication with $|\lambda| = \infty$, and determine which 1–connected simple Lie groups have this property.

We conclude this section by giving our notation and terminology. The usual conventions of homotopy theory will hold. All spaces will be based and will have the homotopy type of CW-complexes. They will be assumed to be nilpotent, and will often be 1–connected. All maps and homotopies will preserve base points. We will not distinguish notationally between a map and its homotopy class, but will refer to an actual map as a function. For spaces $A$ and $X$, we let $[A, X]$ denote the set of homotopy classes from $A$ to $X$. A map (or homotopy class) $f : A \to A'$ determines a function $f^* : [A', X] \to [A, X]$ in the obvious way. Furthermore, $f$ induces a homomorphism of homotopy groups, denoted $f_* : \pi_s(A) \to \pi_s(A')$. 