Hyperbolic heat conduction in the semi-infinite body with a time-dependent laser heat source

M. Lewandowska

Abstract The Cattaneo hyperbolic and classical parabolic models of heat conduction in the laser irradiated materials are compared. Laser heating is modelled as an internal heat source, whose capacity is given by $g(x, t) = I(t)(1 - R)\mu \exp(-\mu x)$. Analytical solution for the one-dimensional, semi-infinite body with the insulated boundary is obtained using Laplace transforms and the discussion of solutions for different time characteristics of the heat source capacity (constant, instantaneous, exponential, pulsed and periodic) is presented.

List of symbols

- $a$ thermal diffusivity = $k/(\rho c_p)$, m$^2$/s
- $B_u$ dimensionless Bouguer number
  $$= \mu_1^1 = \beta \sqrt{\tau/2}$$
- $c_p$ specific heat at constant pressure, J/(kg K)
- $C$ heat capacity, J/(m$^3$K)
- $F_0$ dimensionless Fourier number
  $$= at/x^2 = \tau/(2\pi^2)$$
- $g$ capacity of internal heat source, W/m$^3$
- $G$ electron–phonon coupling constant in the Anisimov model, W/(m$^3$K)
- $I$ laser incident intensity, W/m$^2$
- $I_0$ arbitrary reference laser intensity, W/m$^2$
- $I_0$ modified Bessel function, 0th order
- $k$ thermal conductivity, W/(mK)
- $L$ Laplace operator
- $R$ surface reflectance
- $q$ heat flux vector, W/m$^2$
- $s$ Laplace variable
- $t$ time, s
- $t_1$ duration of laser pulse, s
- $t_k$ relaxation time of heat flux, s
- $T$ temperature, K
- $T_m, T_0$ arbitrary reference temperatures, K
- $w$ speed of heat propagation = $(a/\tau_k)^{1/2}$, m/s
- $x, y, z$ Cartesian coordinates, m
- $X, Y, Z$ dimensionless Cartesian coordinates

Greek symbols

- $\beta$ dimensionless absorption coefficient
- $\gamma, \gamma_m, \gamma_p$ auxiliary coefficients defined by Eqs. (20c), (20d), (20e), respectively
- $\eta$ dimensionless rate of energy absorbed in the medium
- $\mu$ absorption coefficient
- $\theta$ dimensionless temperature
- $\rho$ density
- $\tau$ dimensionless time
- $\tau_i$ dimensionless duration of laser pulse
- $\Phi$ vector of dimensionless heat flux
- $\psi$ dimensionless capacity of internal heat source
- $\psi_0$ constant coefficients related to the dimensionless capacity of internal heat source
- $\omega$ frequency of a periodic heat source

Superscript

- $^*$ transformed variable

Indices

- $H$ related to the hyperbolic model
- $P$ related to the parabolic model
- $e$ related to electrons
- $l$ related to the metal lattice

1 Introduction

An increasing interest has arisen recently in the use of heat sources such as lasers and microwaves, which have found numerous applications related to material processing (e.g. surface annealing, welding and drilling of metals [1, 2], sintering of ceramics [3]) and scientific research (e.g. measuring physical properties of thin films [4], exhibiting microscopic heat transport dynamics [5]). Lasers are also routinely used in medicine.

In many applications it is particularly important to know the temperature distribution inside an impacted medium. To assess theoretically changes in the temperature field inside the body, the heat conduction equation should be solved. The following parabolic model of heat conduction

$$\frac{\partial T}{\partial t} = a\nabla^2 T + \frac{g}{\rho c_p},$$

based on the classical Fourier law, is used in many common engineering problems, but in some situations, e.g.
when extremely short laser pulses or very high frequencies are concerned, it may give inaccurate results.

The Fourier model assumes infinite speed of heat transport, i.e. a local thermal disturbance is instantaneously felt at each point in the medium, which is a non-physical behaviour. More realistic is the following hyperbolic model of heat conduction

\[
t_k \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a \nabla^2 T + \frac{1}{\rho c_p} \left( t_k \frac{\partial g}{\partial t} + g \right) ,
\]  

(2)

resulting from the incorporation of the Catteneo flux law [6]

\[
t_k \frac{\partial q}{\partial t} + q = -k \nabla T .
\]  

(3)

into the energy conservation equation

\[
\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot q + g .
\]  

(4)

In Eqs. (2) and (3) \( t_k \) is the thermal relaxation time characterising a time scale of the heat flux relaxation process and it is related to the speed of propagation of the thermal wave in the medium \( w \) by

\[
t_k = \frac{a}{w^2} .
\]  

(5)

For a material that absorbs laser energy internally, the energy source term in Eq. (2) is modelled as [7–9]

\[
g(x, t) = I(t)(1 - R)\mu \exp(-\mu x)
\]

(6)

where \( I(t) \) is the laser incident intensity, \( R \) is the surface reflectance of the body and \( \mu \) is the absorption coefficient.

This model assumes no spatial variations of \( I(t) \) in the plane perpendicular to the laser beam and no heat transport in the direction perpendicular to the beam. Ready [2] analysed cases in which the problem times are sufficiently small to satisfy the last condition. Several authors have studied analytically the parabolic model of heat conduction with the laser heat source given by Eq. (6) and with a convective boundary condition [8, 9].

According to Eq. (6) approximately 99% of energy is absorbed within a depth of \( \delta = 4.6/\mu \). For metals (\( \mu \) of the order of \( 10^7–10^8 \text{ m}^{-1} \)) almost all energy is absorbed within a depth of the order of 0.1 \( \mu \) which for many purposes can be treated as a skin effect. For this reason some authors simplify the model considering the laser radiation as a heat source which is \( x \)-independent and non-zero only within a thin layer of the body (e.g. [10–12]) or even as a surface heat flux (e.g. [12, 13]). Grigor’ev [7] discusses cases in which the last assumption is justified.

The aim of the present work is to examine the necessity of using the hyperbolic model of heat conduction while modelling heat conduction in a body subjected to a laser heat source with various time characteristics. To study discrepancies between hyperbolic and related parabolic solutions, we analyse the case of heat conduction in the body with the internal heat source whose capacity is given by Eq. (6). In the present work the surface reflectance and the thermophysical parameters of the body are assumed to be constant. We solve analytically, using the Laplace transforms method, a one-dimensional case of the hyperbolic equation of heat conduction [Eq. (2)] for the semi-infinite body with the insulated boundary. The results achieved from the hyperbolic model are compared with those calculated from the parabolic equation of heat conduction [Eq. (1)].

2

Model

2.1

Hyperbolic equation in dimensionless coordinates

The following dimensionless variables are defined:

\[
X = \frac{wx}{2a} \quad (7a)
\]

\[
Y = \frac{wy}{2a} \quad (7b)
\]

\[
Z = \frac{wz}{2a} \quad (7c)
\]

\[
\tau = \frac{t}{2t_k} \quad (7d)
\]

\[
\theta = \frac{T - T_0}{(T_m - T_0)} \quad (7e)
\]

\[
\Phi = \frac{q}{\rho c_p(T_m - T_0)} \quad (7f)
\]

\[
\Psi = \frac{gt_k}{\rho c_p(T_m - T_0)} . \quad (7g)
\]

The dimensionless forms of Eqs. (1)–(4) are as follows, respectively

\[
\frac{\partial \theta}{\partial \tau} = \nabla^2 \theta + 4\psi \quad (8)
\]

\[
\frac{\partial^2 \theta}{\partial \tau^2} + \frac{\partial \theta}{\partial \tau} = \nabla^2 \theta + 2\frac{\partial \psi}{\partial \tau} + 4\psi \quad (9)
\]

\[
\frac{\partial \Phi}{\partial \tau} + 2\Phi = -\nabla \theta \quad (10)
\]

\[
\frac{\partial \theta}{\partial \tau} = -\nabla \Phi + 2\psi . \quad (11)
\]

The dimensionless heat source capacity according to Eq. (6) is

\[
\psi(X, \tau) = \psi_0 \eta(\tau) \exp(-\beta X) \quad (12a)
\]

where

\[
\psi_0 = \frac{I_r(1 - R)\mu t_k}{\rho c_p(T_m - T_0)} \quad (12b)
\]

\[
\eta(\tau) = \frac{I(2t_k \tau)}{I_r} \quad (12c)
\]

\[
\beta = \frac{2w t_k \mu}{} . \quad (12d)
\]

2.2

Formulation of the problem

Further on, we shall consider a one-dimensional case of Eq. (12a) which, after the incorporation of the heat source capacity described by the Eq. (12a), takes the form

\[
\frac{\partial^2 \theta}{\partial \tau^2} + \frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2} + 2\psi_0 \frac{\partial \eta}{\partial \tau} \frac{\eta}{\eta} \exp(-\beta X) . \quad (13)
\]

The dimensionless initial and boundary conditions to be applied to Eq. (13) are as follows

\[
\theta(X, 0) = 0 \quad (14a)
\]

\[
\frac{\partial \theta}{\partial \tau}(X, 0) = 2\psi_0 \eta(0) \exp(-\beta X) \quad (14b)
\]

\[
\frac{\partial \theta}{\partial X}(0, \tau) = 0 \quad (14c)
\]