RESEARCH ARTICLE

Separation Axioms on $\mathbb{N}^\infty$-Systems

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Abstract

Projection algebras (spaces) are nothing but $\mathbb{N}^\infty$-systems. Computer scientists use these algebras for the specification of infinite objects (process) which can not be denoted by finite terms. Using the closure operator given in [9], we consider these algebras as topological spaces and investigate the separation axioms for them. Among other things, we get some equivalent conditions to separatedness defined and studied in [9]. We also study the relations between separatedness and other separation axioms. Finally, we characterize the subdirectly irreducible projection algebras.

Key words: Projection algebra, separation axioms, subdirectly irreducible.

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1. Introduction

Consider the monoid of extended natural numbers $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $m.n = min\{m,n\}$, where $n < \infty$ for every natural number $n$. A left $\mathbb{N}^\infty$-system (act [10]), also called a projection space [5,6,9] and a projection algebra [4,7,8,11], is nothing but a set $X$ together with a map $\lambda : \mathbb{N}^\infty \times X \to X$ such that denoting $\lambda(n,x)$ by $nx$, we have

$$(m.n)x = m(nx) \quad \text{and} \quad \infty x = x$$

for $m, n \in \mathbb{N}^\infty$ and $x \in X$.

Now, the homomorphisms between projection algebras are called projection maps and the resulting category is denoted by PRO.

Remark 1.1. The sub projection algebra generated by $Y \subseteq X$ is $<Y> = \mathbb{N}^\infty Y$. It is easily proved that $x$ is the only generator of the cyclic projection algebra $<x>$.

Recall from [3,9] that $C_s$ defined by

$$\overline{A} = C_s(A) = \{x \in X \mid sx \in A, \forall s \in \mathbb{N}\}$$

for every sub projection algebra $A$ of $X$ gives a subprojection algebra. One can extend the closure operator ([2]) $C_s$ to subsets $Y$ of $X$ by

$$\overline{Y} = C_s(Y) = C_s(<Y>)$$
and get a topology on $X$ whose closed sets are in fact the closed subalgebras.

**Remark 1.2.** Each nonempty projection algebra has a fixed element $a_0$, in the sense that $na_0 = a_0$ for each $n \in \mathbb{N}$. In fact, for each $a$, $1a$ is a fixed element. Notice that for each fixed element $a_0$, $\{a_0\}$ is a subalgebra, and

$$\overline{\{a_0\}} = \{x \in X : sx = a_0, \forall s \in \mathbb{N}\}$$

contains no other fixed element. Thus for distinct fixed elements $a_0, b_0$, $\{a_0\} \neq \{b_0\}$.

Also, notice that projection maps send fixed elements to fixed ones.

**Remark 1.3.** The equivalence relation $\theta$ given by $a \theta b$ if and only if $1a = 1b$ partitions a projection algebra $X$ into blocks of (closed) subalgebras each with only one fixed element. Thus, every projection algebra is a disjoint union (coproduct in PRO) of projection algebras with a single fixed element. So the notions transferable from the factors to coproducts can in fact be studied only for projection algebras with a single fixed element. We will see some instances of this fact in the sequel.

### 2. Separation axioms

In this section we study some of the counterparts of topological separation axioms for projection spaces (algebras) defined in terms of closed subalgebras. We will see that many of them are surprisingly, or interestingly, in some sense trivial or are equivalent.

**Definition 2.1.** A projection algebra (space) $X$ is said to be:

1. $T_0$ if for every pair of distinct elements $x, y$ of $X$ there exists a closed subalgebra $F_x$ of $X$ containing $x$ and not $y$ or there exists a closed subalgebra $F_y$ containing $y$ but not $x$.

2. $T_1$ if for each pair of distinct nonfixed elements $x, y$ of $X$ there exist closed subalgebras $F_x$, which contains $x$ and not $y$, and $F_y$ containing $y$ but not $x$.

3. $T_2$ if for each pair of distinct nonfixed elements $x, y$ there exist closed subalgebras $F_x$, with $x \in F_x, y \notin F_x$, and $F_y$, with $y \in F_y, x \notin F_y$, such that $F_x \cup F_y = X$.

4. $T'_2$ if for all $x \neq y$ in $X$ there are closed subalgebras $F_x, F_y$ such that $x \in F_x, y \in F_y$, with $x \notin F_y$ or $y \notin F_x$, and $F_x \cup F_y = X$. 