On Comparing Two Chains of Numerical Semigroups and Detecting Arf Semigroups

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Abstract

If $T$ is a numerical semigroup with maximal ideal $N$, define associated semigroups $B(T) := (N - N)$ and $L(T) = \cup\{(hN - hN) : h \geq 1\}$. If $S$ is a numerical semigroup, define strictly increasing finite sequences $\{B_i(S) : 0 \leq i \leq \beta(S)\}$ and $\{L_i(S) : 0 \leq i \leq \lambda(S)\}$ of semigroups by $B_0(S) := S = L_0(S)$, $B_{\beta(S)}(S) := \mathbb{N} = L_{\lambda(S)}(S)$, $B_{i+1}(S) := B_i(S)$ for $0 < i < \beta(S)$, $L_{i+1}(S) := L(L_i(S))$ for $0 < i < \lambda(S)$. It is shown, contrary to recent claims and conjectures, that $B_2(S)$ need not be a subset of $L_2(S)$ and that $\beta(S) - \lambda(S)$ can be any preassigned integer. On the other hand, $B_2(S) \subseteq L_2(S)$ in each of the following cases: $S$ is symmetric; $S$ has maximal embedding dimension; $S$ has embedding dimension $e(S) \leq 3$. Moreover, if either $e(S) = 2$ or $S$ is pseudo-symmetric of maximal embedding dimension, then $B_i(S) \subseteq L_i(S)$ for each $i$, $0 \leq i \leq \lambda(S)$. For each integer $n \geq 2$, an example is given of a (necessarily non-Arf) semigroup $S$ such that $\beta(S) = \lambda(S) = n$, $B_i(S) = L_i(S)$ for all $0 \leq i \leq n - 2$, and $B_{n-1}(S) \subsetneq L_{n-1}(S)$.

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1. Introduction

All semigroups considered below are numerical semigroups, that is, submonoids of the natural numbers $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ under addition. We adopt the conventions of [4] and [2]. In particular, except for the degenerate cases $S = \mathbb{N}$ and $S = \{0\}$, we have a canonical form description of a semigroup $S = \langle a_1, \ldots, a_\nu \rangle$ so that $\nu \geq 2$; $a_i < a_{i+1}$ for $1 \leq i < \nu - 1; \text{GCD}(a_1, \ldots, a_\nu) = 1$ (equivalently, $\mathbb{N} \setminus S$ is finite); and $a_i \notin \{a_j : 1 \leq j < \nu, j \neq i\}$ for $1 \leq i \leq \nu$.

Given such a generating set of $S$, we call $\nu$ the embedding dimension of $S$ and denote it by $e(S)$; and we call $a_1$, the least positive element of $S$, the multiplicity of $S$ and denote it by $\mu(S)$. In general, $e(S) \leq \mu(S)$. Besides $\mathbb{N}$, semigroups $S$ satisfying $e(S) = \mu(S)$ are said to be of maximal embedding dimension. Their role in characterizing a class of Noetherian integral domains of maximal embedding dimension is given in [2, Proposition II.2.10].

Let $S$ be a semigroup (other than $\mathbb{N}$, $\{0\}$). The maximal ideal of $S$ is $M(S) := S \setminus \{0\}$. The largest element of $\mathbb{N} \setminus M(S)$ is called the Frobenius number of $S$ and is denoted by $g(S)$. For instance, if $S := \langle 4, 7 \rangle = \langle 4, 7, 11 \rangle$. 


We say that Theorem 2.4(a),(b). conclusion holds if conjecture that produces a semigroup paper by showing that the previous claim for $i$.

However, we can prove this only for $0$.

an arbitrary semigroup $S$ such that

On the other hand, by re-examining the work underlying this conjecture in [2], (fifteen characterizations of Arf semigroups and [2, Proposition II.1.12]). In fact, is of maximal embedding dimension if and only if $B(S) = L(S)$ (see Proposition 2.2(d)).

By iterating the $B$ and $L$ constructions, one arrives at an interesting class of semigroups of maximal embedding dimension called the Arf semigroups. (See [2, Theorem I.3.4] for fifteen characterizations of Arf semigroups and [2, Theorem II.2.13] for their role in characterizing Arf rings, an important class of rings studied in algebraic geometry and commutative algebra: cf. [1], [5]). In general, for any semigroup $S$, we obtain two ascending chains of semigroups

$$B_0(S) := S \subseteq B_1(S) := B(B_0(S)) \subseteq \cdots \subseteq B_{h+1}(S) := B(B_h(S)) \subseteq \cdots,$$

$$L_0(S) := S \subseteq L_1(S) := L(L_0(S)) \subseteq \cdots \subseteq L_{h+1}(S) := L(L_h(S)) \subseteq \cdots.$$ We say that $S$ is an Arf semigroup in case $B_i(S) = L_i(S)$ for each $i \geq 0$. For an arbitrary semigroup $S$, we define $\beta(S)$ and $\lambda(S)$ to be the least integers such that $B_{\beta(S)}(S) = \mathbb{N} = L_{\lambda(S)}(S)$. Of course, if $S$ is an Arf semigroup, then $\beta(S) = \lambda(S)$.

We come now to the focus of this paper. Since any semigroup $S$ satisfies $B_0(S) = L_0(S)$ and $B_{i+1}(S) \subseteq L_{i+1}(S)$, it is tempting to conjecture that $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$ and for arbitrary $S$: “There is much calculational evidence to suggest that $\lambda(S) \leq \beta(S)$, and indeed that $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$.

However, we can prove this only for $0 \leq i \leq 2$.”; cf. [2, p. 14]. We begin this paper by showing that the previous claim for $i = 2$ is mistaken, for Example 2.3 produces a semigroup $S$ such that $B_2(S) \nsubseteq L_2(S)$. In that example, $e(S) = 4$. On the other hand, by re-examining the work underlying this conjecture in [2], we are led to Theorem 2.4(c): if $e(S) \leq 3$, then $B_2(S) \subseteq L_2(S)$. The same conclusion holds if $S$ is symmetric or of maximal embedding dimension: see Theorem 2.4(a),(b).

Furthermore, the semigroup $S$ in Example 2.3 shows also that the conjecture that $\lambda(S) \leq \beta(S)$ is false, as it satisfies $\beta(S) < \lambda(S)$. Moreover, the