RESEARCH ARTICLE

On Generation of $C_0$ Semigroups and Nonlinear Operator Semigroups

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Abstract

Generation of $C_0$ semigroups and nonlinear operator semigroups are proved by a different method, based on the theory of difference equations.

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1. Introduction

Let $\omega \in \mathbb{R}$ and $M \geq 1$. Let $X$ be a real Banach space with the norm $\| \cdot \|$. Let $B: D(B) \subset X \rightarrow X$ be a linear operator, which satisfies (i) and (ii).

(i) $B$ is closed and has domain $D(B)$ dense in $X$.

(ii) The resolvent set $\rho(B)$ of $B$ contains $(\omega, \infty)$ and 

$$\|(I - \lambda B)^{-n}\| \leq M(1 - \lambda \omega)^{-n}$$

holds for $\lambda > 0, \lambda \omega < 1$ and $n = 1, 2, 3, \ldots$.

It is well-known [9], [20] that $B$ generates a $C_0$ semigroup $T(t), t \geq 0,$ $\|T(t)\| \leq M e^{\omega t}$ holds, and $T(t)u_0$ for $u_0 \in D(B)$ is a unique classical solution to the Cauchy problem

$$\frac{d}{dt} u = Bu, t > 0, u(0) = u_0.$$  \hfill (1)

Here by a $C_0$ semigroup $T(t)$, it is meant that $T(t), t \geq 0$ is a family of bounded linear operators on $X$, such that $T(0) = I, T(t+s) = T(t)T(s)$ for $t, s \geq 0,$ and $\lim_{t \to 0} T(t)x = x$ for $x \in X$ hold.

The above result is proved [9], [20] (Page 19) by applying the Hille-Yosida theorem, combined with a renorming technique. This result is due (independently) to Feller, Phillips and Miyadera (see [9] for references to the original works). This result is also valid under the seemingly more general conditions (iii) and (iv):

(iii) Range condition. The range of $(I - \lambda B)$ contains $D(B)$ for small enough $\lambda > 0$ with $\lambda \omega < 1$. 


(iv) \((I - \lambda B)^{-1}x\) is single-valued for \(x \in \overline{D(B)}\), and \(\|u\| \leq M(1 - \lambda \omega)^{-n}\|x\|\) holds for all \(\lambda > 0\) with \(\lambda \omega < 1\), \(x \in D(B)\), and \(u = (I - \lambda B)^{-n}x, n = 1, 2, 3, \ldots\).

For generation of semigroups of nonlinear operators on \(X\), let \(A: D(A) \subset X \rightarrow X\) be a nonlinear multi-valued operator, which satisfies (iii) and (v).

(v) Dissipativity. \(\|u - v\| \leq \|u - v - \lambda(x - y)\|\) holds for all \(\lambda > 0, u, v \in D(A), x \in (A - \omega)u,\) and \(y \in (A - \omega)v\).

It is proved by Crandall-Liggett [5] that \(A\) generates a nonlinear operator semigroup \(T(t), t \geq 0\). When applied to \(u_0 \in D(A)\), \(T(t)u_0\) gives a unique generalized solution to the Cauchy problem

\[
\frac{d}{dt} u \in Au, \ t > 0, \ u(0) = u_0,
\]

the notion of solution being due to Benilan [2]. The generalized solution is a strong one if \(X\) is reflexive [5]. It is also proved in [14] that equation (2) has a strong solution if \(A\) is embeddedly quasi-demi-closed. The notion of embeddedly quasi-demi-closed is weaker than that of continuous or demi-closed.

The proof by Crandall-Liggett [5] uses the method of mathematical induction. This method is also used by Kobayashi [11] (see also Takahashi [21], Miyadera [17]) in his generalization of Crandall-Liggett theorem [5]. In this paper, we provide a different method to solve the problem. Our method is based on the theory of difference equations, which applies to both generation of \(C_0\) semigroups (under (iii) and (iv)) and that of nonlinear operator semigroups. The estimates we derive in Propositions 3, 4 are comparable to [5] (Page 271) and [17] (Page 135) (see Section 3 for details).

More references on this subject can be found in [1], [2], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [17], [18], [19], [20], [21].

2. Some estimates

Define the operator \(A\) as in Section 1, which satisfies (iii) and (v). Let \(J_\mu x \equiv (I - \lambda A)^{-1}x\) for \(\mu > 0\) and \(x \in D(A)\). We need to use the following lemmas.

**Lemma 1.** For \(n \in \mathbb{N}, x \in D(A),\) and \(\omega \geq 0, \mu > 0,\) with \(\mu \omega < 1\), the inequality holds:

\[
\|J_\mu^nx - x\| \leq n\mu(1 - \mu \omega)^{-n}\|Ax\|.
\]

Here \(\|Ax\| \equiv \inf\{y: y \in Ax\}\).

**Proof.** See [5] for a proof.

**Lemma 2.** For \(x \in D(A), n, m \in \mathbb{N}\) and \(\lambda \geq \mu > 0, \omega \geq 0\) with \(\mu \omega, \lambda \omega < 1\), the inequality holds:

\[
a_{m,n} \leq \gamma \alpha a_{m-1,n-1} + \gamma \beta a_{m,n-1}.
\]

Here \(a_{m,n} \equiv \|J_\mu^nx - J_\lambda^nx\|, \gamma \equiv (1 - \mu \omega)^{-1}, \alpha \equiv \frac{\mu}{\lambda},\) and \(\beta \equiv 1 - \alpha\).