RESEARCH ARTICLE

Minimal Presentations and Efficiency of Semigroups

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Abstract

A finite semigroup $S$ is said to be efficient if it can be defined by a presentation $\langle A \mid R \rangle$ with $|R| - |A| = \text{rank}(H_2(S))$. In this paper we demonstrate certain infinite classes of both efficient and inefficient semigroups. Thus, finite abelian groups, dihedral groups $D_{2n}$ with $n$ even, and finite rectangular bands are efficient semigroups. By way of contrast we show that finite zero semigroups and free semilattices are never efficient. These results are compared with some well-known results on the efficiency of groups.

1. Introduction

This paper aims to extend the concepts of efficiency and inefficiency from groups to semigroups.

Let $A$ be an alphabet. We denote by $A^+$ the free semigroup on $A$ consisting of all non-empty words over $A$. Let $\varepsilon$ denote the empty word and let $A^* = A^+ \cup \{\varepsilon\}$. A semigroup presentation is an ordered pair $\langle A \mid R \rangle$, where $R \subseteq A^+ \times A^+$. A semigroup $S$ is said to be defined by the semigroup presentation $\langle A \mid R \rangle$ if $S \cong A^+ / \rho$, where $\rho$ is the congruence on $A^+$ generated by $R$. For $w_1, w_2 \in A^+$, we write $w_1 \equiv w_2$ if $w_1$ and $w_2$ are identical words, and we write $w_1 = w_2$ if they represent the same element of $S$, that is if $(w_1, w_2) \in \rho$.

Let $w_1, w_2$ be words in $A^+$. If there exist $\alpha, \beta \in A^*$ and $(u, v) \in R$ such that $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$, then we say that $w_2$ is obtained from $w_1$ by one application of a relation from $R$. We say that $w_1 = w_2$ is a consequence of $R$ if there exists a sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_k$ such that, for each $i = 1, \ldots, k - 1$, $\alpha_i \equiv \alpha_{i+1}$ is obtained from $\alpha_i$ by one application of a relation from $R$.

The deficiency of a finite semigroup presentation (respectively, group presentation) $\mathcal{P} = \langle A \mid R \rangle$ is defined to be $|R| - |A|$, and is denoted by $\text{def}(\mathcal{P})$ (respectively, $\text{def}_G(\mathcal{P})$). The deficiency of a finitely presented semigroup (respectively, group) $S$ is

$$\text{def}(S) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite presentation for } S \}$$

(respectively, $\text{def}_G(S) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite group presentation for } S \}$). A presentation for which this minimum is attained is called a minimal presentation. It is well known that if $S$ is a finite semigroup (respectively, group) then $\text{def}(S) \geq 0$ (respectively, $\text{def}_G(S) \geq 0$). A better bound for deficiency is obtained by considering the second integral homology of $S$.

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For a semigroup $S$, let $S^1$ denote $S$ with an identity adjoined to it if necessary. A (left) projective resolution of $\mathbb{Z}$ over the monoid ring $\mathbb{Z}S^1$ is an exact sequence
\[
\cdots \rightarrow B_n \xrightarrow{\delta_n} B_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} B_1 \xrightarrow{\delta_1} B_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0,
\]
in which each $B_n$ is a (left) projective $\mathbb{Z}S^1$-module and $\mathbb{Z}$ is a trivial (left) $\mathbb{Z}S^1$-module. By applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}S^1} -$ where $\mathbb{Z}$ is a trivial right $\mathbb{Z}S^1$-module, we obtain the chain complex
\[
\cdots \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}S^1} B_n \xrightarrow{1 \otimes \delta_n} \mathbb{Z} \otimes_{\mathbb{Z}S^1} B_{n-1} \xrightarrow{1 \otimes \delta_{n-1}} \cdots \xrightarrow{1 \otimes \delta_2} \mathbb{Z} \otimes_{\mathbb{Z}S^1} B_1 \xrightarrow{1 \otimes \delta_1} \mathbb{Z} \otimes_{\mathbb{Z}S^1} B_0 \rightarrow 0
\]
of abelian groups. Then the $n$th (left) homology of $S$ (with coefficients in $\mathbb{Z}$) is the abelian group $H_n(S) = \text{ker}(1 \otimes \delta_n)/\text{im}(1 \otimes \delta_{n+1})$.

It is well known that $H_2(S)$ of a finite group $S$ is a finite abelian group and that $\text{def}_G(S) \geq \text{rank}(H_2(S))$. A group is said to be efficient (as a group) if $\text{def}_G(S) = \text{rank}(H_2(S))$, and inefficient otherwise. The efficiency problem for groups has been studied over many years; see [1], [2], [7], [11], [12]. It has recently been shown by S. J. Pride (unpublished) that, for a finite semigroup $S$,
\[
\text{def}(S) \geq \text{rank}(H_2(S)).
\]

Similarly, we call a finite semigroup $S$ efficient if $S$ has a presentation $P = \langle A \mid R \rangle$ such that $\text{def}(P) = \text{rank}(H_2(S))$ and inefficient otherwise.

In this paper, we give examples both of efficient semigroups and of inefficient semigroups. We show that finite abelian groups, dihedral groups $D_{2n}$ ($n$ even) and rectangular bands are all efficient. By contrast, finite zero semigroups and finite free semilattices are inefficient. In the process, we calculate their second homology groups and find minimal presentations for them.

2. Efficiency of groups as semigroups

If $S$ is a group, then every semigroup presentation for $S$ is also a group presentation for $S$. Therefore, if $S$ is efficient as a semigroup, it is also efficient as a group. In this section, we show that the converse holds for finite abelian groups and dihedral groups $D_{2n}$ ($n$ even).

**Theorem 2.1.** Finite abelian groups are efficient as semigroups. More precisely, a finite abelian group has a minimal semigroup presentation
\[
P_S = \langle a_1, \ldots, a_r \mid a_i^{q_i+1} = a_i, \ a_i^{q_i} = a_i^{q_i}, \ a_1a_ja_1^{q_i-1} = a_j, \ a_ka_l = a_la_k \rangle
\]
where $r \geq 1$ and $q_j$ divides $q_{j+1}$ for all $j = 1, \ldots, r - 1$.

**Proof.** A finite abelian group $G$ of rank $r$ can be expressed as a direct product of cyclic groups of orders $q_j > 1$ ($1 \leq j \leq r$) with $q_j$ dividing $q_{j+1}$ ($1 \leq j < r$).