RESEARCH ARTICLE

Spectral Inclusions for Semigroups of Closed Operators

Peer Christian Kunstmann

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Abstract

We show that several spectral inclusions known for \( C_0 \)-semigroups fail for semigroups of closed operators, even if they can be regularized. We introduce the notion of spectral completeness for the regularizing operator \( C \) which implies equality of the spectrum and the \( C \)-spectrum of the generator. We prove spectral inclusions under this additional assumption. We give a series of examples in which the regularizing operator is spectrally complete including generators of integrated semigroups, of distribution semigroups, and of some semigroups that are strongly continuous for \( t > 0 \).

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1. Introduction

Suppose that \((T_t)\) is a \( C_0 \)-semigroup in a Banach space \( X \) with generator \( A \). Then it is well known that the spectral inclusion

\[
\exp(t\sigma(A)) \subseteq \sigma(T_t), \quad t > 0
\]

holds (see e.g. [3, Theorem 2.16]). Here \( \sigma(A) := \mathbb{C} \setminus \rho(A) \) is the spectrum of \( A \) and \( \rho(A) \) denotes the resolvent set of \( A \), i.e. the set of all complex \( \lambda \) such that \( \lambda - A \) has an inverse in \( L(X) \), the space of all linear bounded operators on \( X \).

If \((T_t)\) is just a semigroup of closed operators in \( X \), i.e. \( T_0 = \text{Id}_X \) and \( T_s T_t \subseteq T_{s+t} \) for all \( s, t \geq 0 \), but is regularizable, the generator \( A \) of \((T_t)\) is a closed linear operator in \( X \) (see Section 2) and it is natural to ask whether (1) still holds.

In Theorem 1 of [5], the spectral inclusion (1) was claimed in the case that there is a regularizing operator \( C \) with dense range such that \((T_t C)\) is exponentially bounded in \( L(X) \).

However, since there are well known examples of semigroups of bounded operators that are strongly continuous for \( t > 0 \) whose generators \( A \) have empty resolvent, this claim is false in general. We give an example in 5.1 which also shows that the assertion of Theorem 4 in [5] is false in general.

Hence one needs additional assumptions for (1). In this paper we present conditions on the regularizing operator which are sufficient for (1) to hold. These conditions are mild enough to be satisfied in a number of examples such as for the generators of integrated semigroups, distribution semigroups and the generators of some semigroups that are strongly continuous for \( t > 0 \). The key ingredient is the notion of spectral completeness for \( A \) which we introduce for regularizing operators \( C \). If \( C \) is spectrally complete for \( A \), then \( \sigma(A) \) coincides with the \( C \)-spectrum \( \sigma_C(A) \) of \( A \). This allows to use arguments similar to the case of \( C_0 \)-semigroups and then to
switch from the $C$-spectrum to the spectrum. We conjecture that this can be done in a number of results known for $C_0$-semigroups.

As an example, we present an analogue of a result of G. Greiner (see [10, p. 94]) that characterized the resolvent set of the operators $T_t$ in the case of a $C_0$-semigroup. From this result, which gives also conditions for the inclusion reverse to (1), we derive the spectral inclusion (1). Our results make no assumptions on exponential boundedness and we discuss the assumption of $\text{im } C$ being dense.

The paper is organized as follows. Section 2 contains basic facts for semigroups of closed operators. In Section 3 we introduce the notion of spectral completeness and investigate the relations between the spectrum and the $C$-spectrum of the generator $A$ and the operators $T_t$. In Section 4 we prove the analogue of Greiner’s result from which we derive a characterization of the resolvent set $\rho(T_t)$ in case of a semigroup that is strongly continuous for $t > 0$. This characterization seems to be new even for $C_0$-semigroups. Finally, Section 5 contains our results on spectral inclusions and a discussion on the fine structure of the spectrum.

Throughout this paper $X$ will denote a complex Banach space.

2. Semigroups of closed linear operators

Let $(T_t)_{t \geq 0}$ be a semigroup of closed linear operators in $X$, i.e. a family of closed linear operators satisfying $T_0 = I_X$ and $T_s T_t \subset T_{s+t}$. Then $(T_t)$ is a semigroup of unbounded operators in the sense of [6] if

$$D := \{ x \in \bigcap_{s,t \geq 0} D(T_s T_t) : t \mapsto T_t x \in C([0, \infty), X) \} \neq \{0\}.$$ 

Our main assumption on the semigroup is that $(T_t)$ can be regularized, i.e., that there is an injective operator $C \in L(X)$ such that

$$\text{im } C \subset D \quad \text{and} \quad C^{-1} T_t C = T_t \quad \text{for all } \ t \geq 0. \quad (2)$$

We call such an operator $C$ a regularizing operator. Then $(S_t) := (T_t C)$ defines a $C$-regularized semigroup, i.e. $S_0 = C$ is injective, $S_s S_t = C S_{s+t}$ for all $s,t \geq 0$, and $t \mapsto S_t x \in C([0, \infty), X)$ for all $x \in X$. This notion has been introduced in [2], for a survey see [4]. The existence of a regularizing operator implies $D \neq \{0\}$ and $D(T_s T_t) = D(T_{s+t}) \cap D(T_t)$ for $s,t \geq 0$. The construction of the infinitesimal generator in [6] is based on exponentially bounded trajectories which might not exist (see [6, 3.13]). We proceed in a different way and define the operator $A_0$ via

$$(x, y) \in A_0 \iff x, y \in D \quad \text{and} \quad \lim_{t \to 0} (T_t x - x)/t = y.$$ 

Observe that $C$ commutes with $A_0$. In general the operator $A_0$ is not closed, and even its closure might not be the "right" operator to work with (see [8]). We define the generator of $(T_t)$ to be the operator $A := C^{-1} A_0 C$, i.e.

$$(x, y) \in A \iff (C^* x, C^* y) \in A_0.$$ 

Then $A$ is also the generator of $(S_t)$, i.e. $(x, y) \in A \iff \lim_{t \to 0} (S_t x - C x)/t = C y.$ By [4] $A = C^{-1} AC$. The regularized semigroup $(S_t)$ is the main technical ingredient in our proofs.