RESEARCH ARTICLE

**TK-operator Semigroups for Cryptogroups***

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1. Introduction

Completely regular semigroups, which are unions of their subgroups, are very important and have been studied intensely. *Cryptogroups* are completely regular semigroups on which the Green relation $\mathcal{H}$ is a congruence. It is well known that any cryptogroup is a band of groups and conversely (see [5]).

The most efficient method of studying congruences on regular semigroups is the kernel-trace approach. This method proved quite successful for inverse semigroups (see [6]). Though the analysis for general regular semigroups encounters considerable difficulties (see [2] and [3]), this approach has been established and helpful for the studies of lattices of some completely regular semigroup varieties (see [4]).

For a regular semigroup $S$, denote by $E(S)$ the set of idempotents in $S$ and by $C(S)$ the *lattice of congruences* on $S$. For any $\rho$ in $C(S)$,

$$\ker \rho = \{a \in S \mid (\exists e \in E(S)) a pe\}, \quad \text{tr } \rho = \rho|_{E(S)}$$

are the *kernel* and *trace* of $\rho$, respectively. The fundamental result here is that $\rho$ is uniquely determined by the congruence pair $(\ker \rho, \text{tr } \rho)$. By $\rho K$ and $\rho k$ [resp. $\rho T$ and $\rho t$] we denote the greatest and the least congruences on $S$ having the same kernel [resp. trace] as $\rho$. Thus we obtain four *operators* on $C(S)$, which are denoted by $K, k, T$ and $t$. Let $\Gamma = \{K, k, T, t\}$, we denote by $\Gamma^+$ and $\Gamma^*$, respectively, the free semigroup and free monoid generated by $\Gamma$. For any $\rho \in C(S)$, if we act on $\rho$ by $\Gamma^*$, we can obtain a *network of congruences* $\rho, \rho K, \rho k, \rho T, \rho t, ...$, ordered by inclusion. Considering networks for all congruences on $S$, we may figure out the lattice of congruences on $S$. The research of networks of congruences is a further development of the kernel-trace approach. Several works have been completed, see [7] and [8] for details. In the case of *Clifford semigroups* (semilattices of groups) and *completely simple semigroups*, Petrich determined in [7] and [8] the semigroups generated by these four operators with relations valid in all networks of these kinds of semigroups and thus characterized the entire networks of congruences. These successful works are due to the relatively precise descriptions of congruences on Clifford and completely simple semigroups. We shall manage to do the same work as in [7] and [8] in this paper for cryptogroups, and confirm Petrich's guess that the same analysis of congruences on more general classes of semigroups could be made.

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The special relations among cryptogroup congruences, band congruences and idempotent separating congruences on regular semigroups obtained in [1] give a characterization of cryptogroups in terms of congruences. Section 2 presents this information and other necessary notation and terminology. Using the result for cryptogroups stated above, we obtain in Section 3 the conclusions about the operators similar to that for Clifford semigroups in [7]. As in [7], we determine the semigroup generated by operators $\Gamma$ with relations valid in all networks on general cryptogroups. Furthermore, we obtain the semigroup generated by these operators for $E$-unitary cryptogroups. In Section 4, we investigate further the case of completely simple semigroups. The semigroups generated by these operators for some special classes of completely simple semigroups are offered. In each of these cases, we present an example of a completely simple semigroup.

2. Preliminaries

We shall use the notation and terminology of [2] and [7]. Let $S$ be a regular semigroup. Recall from [2] that for any congruence $\rho$ on $S$, $\rho$ is uniquely determined by its kernel and trace in the following way. An equivalence $\xi$ on $E(S)$ is normal if $\xi = \text{tr} \xi^*$, where $\xi^*$ is the congruence on $S$ generated by $\xi$. A subset $N$ of $S$ is normal if $N = \ker \pi_N$, where $\pi_N$ is the greatest congruence on $S$ which saturates $N$.

A pair $(N, \xi)$ is a congruence pair for $S$, if the following conditions are satisfied.

1. $N$ is a normal subset of $S$,
2. $\xi$ is a normal equivalence on $E(S)$,
3. $N \subseteq \ker (\mathcal{L}\xi\mathcal{L} \cap \mathcal{R}\xi\mathcal{R})^0$,
4. $\xi \subseteq \text{tr} \pi_N$,

where $\gamma^0$ is the greatest congruence on $S$ contained in $\gamma$. In such a case, $\rho_{(N, \xi)}$ is defined by

$$\rho_{(N, \xi)} = \pi_N \cap (\mathcal{L}\xi\mathcal{L} \cap \mathcal{R}\xi\mathcal{R})^0.$$ 

**Result 1.** [2, Theorem 2.13] Let $S$ be a regular semigroup. If $(N, \xi)$ is congruence pair for $S$, then $\rho_{(N, \xi)}$ is the unique congruence $\rho$ on $S$ for which $\ker \rho = N$, $\text{tr} \rho = \xi$. Conversely, if $\rho$ is a congruence on $S$, then $(\ker \rho, \text{tr} \rho)$ is a congruence pair for $S$ and $\rho = \rho_{(\ker \rho, \text{tr} \rho)}$.

Denote by $\mathcal{C}_p(S)$ the poset of all congruence pairs under the componentwise inclusion order. Then the mappings

$$\rho \mapsto (\ker \rho, \text{tr} \rho), \quad (N, \xi) \mapsto \rho_{(N, \xi)}$$

are mutually inverse isomorphisms between $\mathcal{C}(S)$ and $\mathcal{C}_p(S)$. In view of this fact, if we want to act on $\rho_{(N, \xi)}$ by operators, we sometimes act on its corresponding congruence pair $(N, \xi)$ in an obvious way. For any congruence $\rho$ on a regular semigroup, define

$$K_\rho = \{ \gamma \in \mathcal{C}(S) \mid \ker \gamma = \ker \rho \}, \quad T_\rho = \{ \gamma \in \mathcal{C}(S) \mid \text{tr} \gamma = \text{tr} \rho \}.$$ 

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