Project scheduling with irregular costs: complexity, approximability, and algorithms

Alexander Grigoriev¹, Gerhard J. Woeginger²

¹ Department of Quantitative Economies, University of Maastricht, P.O. Box 616, 6200 MD Maastricht, The Netherlands (e-mail: a.grigoriev@ke.unimaas.nl)
² Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands (e-mail: gwoegi@win.tue.nl)

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Abstract. We address a generalization of the classical discrete time-cost tradeoff problem where the costs are irregular and depend on the starting and the completion times of the activities. We present a complete picture of the computational complexity and the approximability of this problem for several natural classes of precedence constraints. We prove that the problem is NP-hard and hard to approximate, even in case the precedence constraints form an interval order. For precedence constraints with bounded height, there is a complexity jump from height one to height two: For height one, the problem is polynomially solvable, whereas for height two, it is NP-hard and APX-hard. Finally, the problem is shown to be polynomially solvable if the precedence constraints have bounded width or are series parallel.

1 Introduction

Due to its practical importance, the discrete time-cost tradeoff problem for project networks has been studied in various contexts by many researchers over the last fifty years; see Kelley & Walker (1959) for an early reference. The modern treatment of this problem started with the dynamic programming approaches of Hindelang & Muth (1979) and Robinson (1975), and with an enumeration algorithm by Harvey & Patterson (1979). An up-to-date overview on the discrete time-cost tradeoff problem is Chapter 4 of the survey by Brucker, Drexl, Möhring, Neumann & Pesch (1999) or Chapter 8 of the book by Demeulemeester & Herroelen (2002). In this paper, we look
at a generalization of the classical discrete time-cost tradeoff problem where
the costs depend on the exact starting and completion times of the activities.

**Statement of the problem.** Formally, we consider instances that are called
projects and that consist of a finite set \( A = \{ A_1, \ldots, A_n \} \) of activities
together with a partial order \( \prec \) on \( A \). All activities are available for processing
at time zero, and they must be completed before a global project deadline \( T \).
Hence, the set of possible starting and completion times of the activities is
\( \{0, 1, \ldots, T\} \). The set of intervals over \( \{0, 1, \ldots, T\} \) (the so-called realizations
of the activities) is denoted by \( R = \{(x, y) \mid 0 \leq x \leq y \leq T\} \). For every
activity \( A_j \), there is a corresponding cost function \( c_j : R \to \mathbb{R}^+ \cup \{\pm \infty\} \)
that specifies for every realization \( (x, y) \in R \) a non-negative cost \( c_j(x, y) \)
that is incurred when the activity is started at time \( x \) and completed at time \( y \).
A realization of the project is an assignment of the activities in \( A \) to the
intervals in \( R \). A realization is feasible if it obeys the precedence constraints:
For any \( A_i \) and \( A_j \) with \( A_i \prec A_j \), activity \( A_j \) is not started before activity \( A_i \)
has been completed. The cost of a realization is the sum of the costs of all activities in this realization. The goal is to find a feasible realization of minimum cost. This problem is called **min-cost** project scheduling with irregular costs, or min-cost PSIC for short.

A closely related problem is **max-profit** project scheduling with irregular
costs, or max-profit PSIC for short. Instead of cost functions \( c_j \) for activity
\( A_j \), here we have profit functions \( p_j : R \to \mathbb{R}^+ \cup \{\pm \infty\} \) that specify for
every realization of \( A_j \) the resulting profit. The goal is to find a feasible
realization of maximum profit. Such a profit may for instance represent
the cost reduction for the project, if a deadline is stretched and an activity
becomes less urgent. Clearly, the min-cost and the max-profit version are
polynomial time equivalent: The transformations \( c_j := \text{const}_1 - p_j \) and
\( p_j := \text{const}_2 - c_j \) with sufficiently large constants \( \text{const}_1 \) and \( \text{const}_2 \) translate
one version into the other. However, the two versions seem to behave quite
differently with respect to their approximability.

**Special cases and related problems.** Various special cases arise if the cost
and profit functions satisfy additional properties. A cost function \( c \) is **monotone**, if
\( [x_1, y_1] \subseteq [x_2, y_2] \) implies \( c(x_1, y_1) \geq c(x_2, y_2) \). A profit function
\( p \) is **monotone**, if \( [x_1, y_1] \subseteq [x_2, y_2] \) implies \( p(x_1, y_1) \leq p(x_2, y_2) \). The
intuition behind these concepts is that short and quick executions should be
more expensive than long and slow executions. It is readily seen that the
general version of PSIC is equivalent to the monotone version with respect
to computational complexity and approximability.

Another interesting special case arises, if \( y_1 - x_1 = y_2 - x_2 \) implies
\( c(x_1, y_1) = c(x_2, y_2) \) and \( p(x_1, y_1) = p(x_2, y_2) \). In this special case, the
cost and the profit of an activity only depend on the length of its realization.