Numbers Rule: The Vexing Mathematics of Democracy, from Plato to the Present
by George G. Szpiro


REVIEWED BY RICHARD F. POTTHOFF

Mainly unadorned with technical details and aimed at a broad audience, this book discusses mathematical aspects of voting and elections, which, to the uninformed, may be surprisingly and unexpectedly convoluted. The book “is by no means a textbook but may serve as accompanying literature for a more rigorous course in political science, economics, administration, philosophy, or decision theory” (p. x). It includes extensive biographical and historical material.

The author has a far-reaching background. His works include books on Kepler’s conjecture, Poincaré’s conjecture, and options pricing. Born in Vienna and educated at the Swiss Federal Institute of Technology (mathematics and physics), Stanford University (M.B.A.), and the Hebrew University of Jerusalem (Ph.D. in finance and mathematical economics), George Szpiro served as a management consultant for McKinsey & Co. and then taught and engaged in research at the Wharton School of the University of Pennsylvania, the University of Zurich, and the Hebrew University. More recently, he has been with a Swiss newspaper as a long-time correspondent in Israel, and as a science reporter. For his mathematics columns he received the 2006 Media Prize from the German Association of Mathematicians.

The chapters of Numbers Rule observe a mostly chronological order with each chapter focusing on one or more persons, as follows:

Chapter 1: The Anti-Democrat (Plato)
Chapter 2: The Letter-Writer (Pliny the Younger)
Chapter 3: The Mystic (Ramon Llull)
Chapter 4: The Cardinal (Cusanus)
Chapter 5: The Officer (Borda)
Chapter 6: The Marquis (Condorcet)
Chapter 7: The Mathematician (Laplace)
Chapter 8: The Oxford Don (Dodgson, a.k.a. Lewis Carroll)
Chapter 9: The Founding Fathers (Alexander Hamilton, Thomas Jefferson, George Washington; also John Quincy Adams, Daniel Webster)
Chapter 10: The Ivy Leaguers (Willcox, Hill, Huntington, others)
Chapter 11: The Pessimists (Arrow, Gibbard, Satterthwaite)
Chapter 12: The Quotarians (Balinski and Young)
Chapter 13: The Postmoderns (miscellany)

After an initial chapter that examines Plato’s ideas and philosophy concerning government, the rest of the book covers two main topics related to voting and elections. The one that receives less space is the subject of Chapters 9, 10, 12, and part of 13, and is about allocating legislative seats to geographic areas based on population, or (what is largely the same problem mathematically) allocating seats to political parties based on votes received in a proportional representation election system.

Chapters 2 through 8, 11, and part of 13 concern the important but subtly difficult problem of how to elect a (single) winner from among three or more competing candidates (or other options). The book is rightly critical of the long-established and widely used plurality or first-past-the-post system, under which each voter votes for one candidate and whoever has the highest number of votes wins—even if it is less than half the total. (The reader needs to be cautioned that the book refers to this system as “majority” rather than “plurality” throughout; different writers may use one term rather than the other [1, pp. 349, 457].) In the United States, there has been much recent controversy concerning such matters as accuracy of vote counts, vote fraud, money in campaigns, and barriers to voting. Important as those issues are, arguably the elephant in the room (unnoticed except in a few quarters, such as the FairVote organization) is the plurality system itself. A major benefit of the book is the attention it draws to the bugs in plurality voting.

All tenable voting systems are essentially the same, of course, for the simple case where only \( n = 2 \) candidates are in the race: Each voter votes for one of the two, and whoever has more votes wins. But even where only two candidates compete under the plurality system, its indirect effects may have deterred viable additional candidates from entering the contest.

Chapter 2 of Numbers Rule describes a murder trial, presided over by Pliny, in which there were three options: execution of the accused, banishment, or acquittal, favored by roughly 30%, 30%, and 40%, respectively, of the senators sitting in judgment. The plurality system that was in effect would have resulted in acquittal had all senators voted their true preferences. But the banishment option won when the execution proponents voted for banishment so as to prevent acquittal from winning. Although strategic voting often produces undesirable results, in this case it could be considered beneficial in the sense that the winner (banishment) was preferred to each of the other two options, by 70% to 30% over execution and by 60% to 40% over acquittal. Although a runoff election between the top two options receives mention (p. 29), though only in a potential vein (such a system was not known at the time), it would have forced a choice between execution and acquittal had banishment, the compromise option, received the fewest votes at the start.

The voting systems of both Borda and Condorcet receive Szpiro’s extensive attention. Both, as was discovered only recently, are essentially reinventions of earlier proposals, of Cusanus and Llull, respectively.

Under the Borda system, each voter ranks the \( n \) (>2) candidates. Scores of \( n, (n - 1), \ldots, 1 \) are assigned to the
candidates who, on a given ballot, are ranked highest, second highest, ..., lowest, respectively. The candidate with the highest total score across all ballots is the winner.

A major argument against the Borda system, advanced by both Condorcet and Laplace, is that it is highly vulnerable to manipulation through strategic voting. For example, if the true order of preferences among three candidates X, Y, and Z is XYZ for 45 voters, YZX for 40, and ZXY for 15, then X is the Borda winner if everyone votes sincerely, but Y wins if 21 of the 40 voters in the second group manipulate by marking their ballots YZX rather than YXZ. Another argument against Borda is that even a candidate who receives more than half of all first-place votes may fail to be the Borda winner, as occurs (but is not explicitly noted) in an example on pages 67–68 and in one on pages 90–91.

Under the Condorcet system, the voter also ranks the n candidates, or, equivalently, makes a choice between the candidates in each of the n(n−1)/2 candidate pairs. (The two are equivalent assuming transitivity, so that a voter who chooses X over Y and Y over Z also chooses X over Z.) In each candidate pair, one finds which contender is ranked by more voters above the other. The Condorcet winner of the election is then the candidate who wins the pairwise contest against each of the (n−1) other competitors. In the Pliny example, banishment is the Condorcet winner. In general, the plurality, Borda, and Condorcet winners can all be different.

A Condorcet winner may not exist, however: What is called the Condorcet paradox may rear its ugly head. As an extreme example, if one third of the electorate provides each of the rankings XYZ, YZX, and ZXY, then X beats Y, Y beats Z, and Z beats X, each by a 2-to-1 margin. If a winner is lacking because there is no Condorcet cycle, then a means of resolution, sometimes called a Condorcet completion method, is needed.

One such method was suggested by Dodgson (described, although somewhat incompletely, on pp. 114–115), but it was found more than a hundred years later to require computation although somewhat incompletely, on pp. 114–115), but it was called a Condorcet completion method, is needed.

The belief that a Condorcet winner rarely exists results in Numbers Rule being overly pessimistic regarding the feasibility of finding satisfactory voting systems. The “impossibility theorem” of Arrow ruling out the existence of systems that satisfy certain apparently reasonable requirements, and the Gibbard-Satterthwaite results showing unpreventability of strategic manipulation in voting systems, are indeed discouraging. But the effect of the Arrow theorem is tempered if cycles do not often occur in practice [4, pp. 4–5, 193].

The mathematically convenient but unrealistic “impartial culture model” (although not mentioned in Numbers Rule) has fostered the impression that real-world cycles are far more common than is actually the case [4]. That model, which has received much attention, rests on the empirically dubious assumption that each voter is drawn independently from a distribution in which each of the n! preference orders is equally likely. Yet for n = 3 (perhaps the most important case practically), even this model produces probabilities of a cycle that are low, though not negligible. With n = 3 candidates and 3 voters, one easily finds that 12 of the (3!)3 = 216 possible outcomes are cyclical, for a probability of 12/216 = .056. The limiting probability of a cycle as the number of voters becomes large, for n = 3, is 1 − (3 × .304) = .088, where .304 is the probability that a majority prefers any given candidate (X, e.g.) pairwise to each of the others (Y and Z), obtainable as the probability, equal to .25 + (sin−1ρ)/(2π) with ρ = 1/3, that two bivariate normal variables with zero means, unit variances, and correlation coefficient 1/3 are both positive. With increasing n, though, the probabilities of no Condorcet winner do increase steadily.

There are, of course, many voting systems for single-winner elections besides plurality, Borda, and Condorcet. Among those others that the book covers are two that are discussed briefly at the end of the last chapter. The first of these is the single transferable vote, also referred to as the alternative vote, instant-runoff voting, or the single-winner version of the Hare method, which is championed by FairVote. An example of this system, although not labeled as such, is the one on pages 105–106. It is used in several countries for certain nationwide elections as well as in scattered