How Mathematicians Think: Using Ambiguity, Contradiction, and Paradox to Create Mathematics

by William Byers


REVIEWED BY EMILY GROSHOLZ

When we learn mathematics from textbooks, as deductively ordered content, the subject matter appears to be timeless; but if by contrast we look at what mathematicians actually do, we see that the formulation and solution of problems, and the discrimination of new ideas, is in fact historical, a dynamic process. Mathematical knowledge is always in flux, and the things of mathematics tend to show up embedded in problems; mathematics takes place at the boundary of the known and the unknown. This is the main insight of William Byers in his book How Mathematicians Think. It leads him to look into the strategies that mathematicians employ as their research drives mathematics forward, and to criticize the twentieth-century philosophy of mathematics, which emphasizes the logical structure of mathematics at the expense of other important aspects.

The book has three sections. The first section investigates how mathematicians exploit ambiguity in their research, both when it merely involves polysemy (as when the circle comes to be associated with both a geometrical curve and an algebraic equation in Descartes’s Géométrie) and when it involves the confrontation of contradiction and paradox (as when mathematicians treat the infinitesimal in the seventeenth century and the infinite in the nineteenth). The second section catalogues some “great ideas” that have inspired mathematical research: the number 1, the notions of equivalence relation and quotient space, the exponential function, continuity, linear approximation, the power set operation, randomness, fractals, and complex numbers. The final section, inspired in part by the author’s study of Zen Buddhism, considers mathematics as an aspect of the human quest for meaning. Byers reminds us that, despite the power of the new technology that mathematical logic and its central notion of computability bequeathed to us in the twentieth century, mere calculation is not creativity. Mathematical research is characterized by growth, change, and the emergence of the unexpected, the discovery of complexity in the apparently simple, and the apprehension of underlying simplicity in the organization of the complex. The model for mathematical research is not Aristotle’s syllogism (wonderful as that invention was), but Plato’s ambiguous and endless dialectic, always ascending the steeps of the Divided Line toward the realm of the Ideas, only to be plunged—usually via a myth—back down to the lowlands again, to the level of dream and shadow.

In Chapter 1, Byers opens the discussion of ambiguity in mathematics with a wide variety of examples. Deductive logic in the favored guise of first-order predicate logic is the preferred instrument of philosophers of mathematics for exhibiting and analyzing mathematical rationality; but it has various limitations. First of all, it is a logic, and mathematical logic is only one branch of mathematics, differing greatly in object and method from other branches. Second, its valid deductive argument forms do not capture the notion of relevance between premises and conclusion properly, in part because it is a logic of propositions rather than a logic of terms, and in part because of (almost inescapable) conventions governing the truth table for the horseshoe operator, “if...then.” So, too, predicate logic cannot express well the systematic relations among the items referred to by its terms, especially certain kinds of taxonomies. It cannot express polysemy: one individual constant must always refer to one and the same thing in the universe of discourse, and the universe of discourse is a set of discrete items cheek by jowl in no special order. After a collection of propositions has been axiomatized, we can reason downward to the theorems that follow deductively from the axioms; but we can’t reason sideways (to assumptions outside the system) or upward (to assumptions that might undergird the axioms themselves). And we can’t ask “contrastive questions,” that is, we can’t ask, by characterizing a situation in a certain way, what other kinds of characterization we are ruling out, alternatives that may help to constitute the meaning of the things we are talking about. Finally, of course, like all logics, first-order predicate logic abhors a contradiction; and, abhorring the infinite as well, it only tolerates the finitary bits of the countably infinite.

Byers thus discusses a variety of cases that elude the expressive powers of the logic favored by philosophers. He notes, for example, the way that the tension, and affinity, between geometry and arithmetic (the Eve and Adam of mathematics) showed itself early on. If we think of a segment of the number line as marked off in units, the relation between the line and the natural numbers seems straightforward. But if we take three line segments to construct a right triangle with two sides that are each measured by 1 unit, we run into trouble with the remaining side, the hypotenuse: what number measures that one? (pp. 35-39). The Pythagorean theorem tells us it must be the square root of 2, and a brief reductio ad absurdum argument proves that this number can’t be the ratio of two natural numbers. Thinking about a line segment in terms of both geometry and arithmetic forces us to revise the notion of number. Indeed, in ensuing centuries this polysemy motivates the novelties of negative, algebraic, and transcendental numbers and breathes new life into complex numbers, as (pp. 39-51) it helps to precipitate analytic geometry, the infinitesimal calculus, and complex analysis.

Byers goes on to discuss the confrontation of mathematicians with apparent contradictions: the treatment of zero at the end of Chapter 2, and the treatment of infinity in...
Chapters 3 and 4. The Greeks never thought of zero as a number; however, some time between the first and fifth centuries CE, mathematicians in India developed a (decimal) positional numeral system. Byers observes that, like an abacus, such a system allows for the possibility of an empty column. Since a notation is needed to mark an empty column (to keep the positions clear), by the fifth century the mark for zero was commonly used, and by the seventh century the astronomer Brahmagupta had spelled out rules governing the use of zero as a number. Byers adds, following John Barrow in The Book of Nothing, that Indian metaphysics was friendlier to nothingness than Greek metaphysics: the Indian notion is ambiguous, both something and nothing, presence and absence, capable of articulation, and so lends itself to a positive treatment of zero. The improved status of zero as a number, Byers goes on to argue, plays an important role in the development of the calculus, where it helps to support speculation about infinitesimals concepts stable enough to be represented and to figure in calculations and equations (pp. 99-109).

If zero is a respectable number, then dividing a natural number by zero \((n/0)\) naturally provokes speculation about whether infinity can be meaningfully represented and can play a useful role in mathematics. Chapters 3 and 4 are devoted to the emergence of infinity in modern mathematics and the fruitfulness of addressing apparent paradoxes in mathematics, rather than simply trying to stay away from them. A paradox, when it first appears, seems to be nothing more than a contradiction and so blocks further discourse. However, if a paradox is viewed in the softer light of ambiguity, Byers argues, it may prove to be very fruitful, as the early development of the study of infinite series shows (pp. 117-125 and pp. 131-145). For example, if one thinks that the infinite and the finite can never consort, then the expression

\[
1 = 1/2 + 1/4 + 1/8 + 1/16 + \ldots
\]

appears quite paradoxical, especially with those three fishy dots, leading where? On the other hand, the mathematicians who learned how to tolerate the conjunction of the finite and infinite, and those dots stretching away into the distance, discovered a whole new subject matter in which such expressions were for the most part deeply meaningful and applicable to other problems in other areas of mathematics. Moreover, when those expressions failed to be meaningful, the investigation of the failure turned out to be fruitful as well: it led to the distinction between convergent and divergent series, and later to the insight that not only a number but also a function can be expressed as an infinite series. The subsequent investigation of Taylor series led further to the concepts of analytic function and radius of convergence: a function is analytic when, at every point, its power series has a radius of convergence that is positive.

In Chapter 4, Byers shows how the notion of infinity helps to organize projective geometry, and allows the operation of forming the power set (the set of all subsets of a given set) in set theory to precipitate a whole hierarchy of infinities. Once again, paradox is transformed into a source of knowledge: the part can indeed be equal to the whole, as long as we reconsider the notion of equality. Who imagined, until Cantor’s wonderful and simple proof, that there could be more than one kind of infinity? But there they are: all those alephs. Or who thought that various familiar sets could be sorted by such infinities? The integers and the algebraic irrationals are countable, but the set of all irrationals is not (because the set of all transcendentals isn’t), and if we think of a line segment, or the Cantor set, as a set of numbers, they aren’t either (pp. 157-181).

Byers then turns his attention to “great ideas” in Chapters 5, 6, and 7. Great ideas, as he shows in Chapter 5, help to organize domains and even to organize knowledge across domains. One of the most interesting examples he mentions is that of the modern reconceptualization of equality as the relation of equivalence \(R\). Members of a set \(X\) are equivalent when three conditions hold: reflexivity or \(xRx\), symmetry or \(yRx\) implies \(yRx\), and transitivity or if \(xRy\) and \(yRz\), then \(xRz\) (pp. 209-218). Designate all the members of \(X\) that are equivalent to \(x\) by \([x]\); the relation \(R\) institutes a partition of \(X\), thereby conferring internal structure on it, as well as creating a whole raft of new objects, the equivalence classes of \(X\). When the set \(X\) is more than a set, that is, when it has distinctive structure and the equivalence relation is extended to express that structure, the consequences can be very deep. A group, for example, may be articulated in terms of a quotient group, as when we “mod out” the integers \(Z\) by a prime \(p\) to produce the finite field (which is thus also a group) \(\mathbb{Z}_p\). Likewise, a topological space may be associated with a quotient space, as when we sort out points on the real line by the equivalence relation “differing by an integer,” that is, \(xRy\) means \(x - y = n\) for some integer \(n\). This equivalence relation produces the interval \([0,1]\) with 0 and 1 identified (which is topologically a circle) as the quotient space; we can use it to show that the real line is a covering space for the circle. So we arrive at different notions of equivalence or isomorphism that apply to different kinds of mathematical structure: isometry for metric spaces, homeomorphism for topological spaces, bijective homomorphism for groups and rings. With this kind of organization, the notion of invariant also arises, that is, a feature that characterizes all members of an equivalence class. One of the most surprising results in modern topology is the way that algebraic structures (such as homotopy groups or homology groups) themselves turn out to be useful invariants for classifying topological spaces.

In Chapter 6, Byers argues that the soul of mathematics lies first of all in the play of ideas and only secondarily in formal proof. To make his point, he reminds us that the things of mathematics are problematic: “Classifying ideas as true or false is just not the best way of thinking about them. Ideas may be fecund; they may be deep; they may be subtle; they may be trivial. These are the kinds of attributes we should ascribe to ideas.” And he quotes David Bohm’s remark that although theories may sometimes be locally clear, they may also become unclear when we try to extend them to other domains (p. 256). The real questions to ask about ideas are: How do they organize the situations in which they appear? And what fruitful quandaries do they lead us into? He then reminds us that logic itself is an idea