The Fourier Method in Russia Before and After V. A. Steklov

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On the occasion of the 150th anniversary of V. A. Steklov’s birth

The Fourier method is one of the techniques widely used for solving boundary value and initial-boundary value problems arising in the theory of partial differential equations. (One finds its more or less detailed presentation in every textbook on mathematical physics; see [1, 35, 36] for the most recent ones.) Cornerstones in this field were laid in the fundamental studies of D. Bernoulli, L. Euler, J. Fourier, S. D. Poisson and G. Lamé. Since their work, the method profoundly influenced many areas of mathematics and its influence was formative for some of them. Moreover, the method is extensively used in applied research. However, its rigorous justification remained open until the end of the 19th century. It was the outstanding Russian mathematician Vladimir Andreevich Steklov who initiated studies of this topic in 1896 (see [31], where the first results about the method’s justification were obtained under rather restrictive assumptions), and his work continued for almost 2 decades (the most general results appeared in [30]).

The 150th anniversary of Steklov’s birth fell on 9 January 2014. All over the world, researchers in all areas of mathematics know his name. Indeed, the widely known mathematical institutes of the Russian Academy of Sciences in Moscow and St. Petersburg are named after Steklov. This commemorates the fact that he was the founding father of their predecessor: the Physical-Mathematical Institute established in 1921 in starving Petrograd (the Civil War still lasted in some corners of what would become the USSR the following year). Steklov was the first director of the Institute until his unexpected and untimely death on 30 May 1926. A biographical sketch of Steklov can be found in [15] along with some recent developments in the study of the Steklov eigenvalue problem, which is an exciting and rapidly developing area on the interface of spectral theory, geometry, and mathematical physics. Further details concerning Steklov’s legacy in mathematical physics are presented in [14].

The traditional approach to justifying the Fourier method requires one to prove uniform convergence of several series. One of the series provides the formal problem’s solution, and the others are obtained by its termwise differentiation as many times as necessary. Let us quote Steklov himself ([34], p. 224):

The necessity to prove the uniform convergence of the series under consideration follows from the essence of the Lamé–Fourier (Euler–Bernoulli) method; indeed, it gives a solution in the form of an infinite series and it is impossible to find its sum or to transform it so that its derivatives can be obtained in a closed form.

1Under this name the method appears in all research papers and textbooks written in Russian including the recent lectures [1] by V. I. Arnold. In textbooks written in English, the method is mostly referred to as separation of variables (see, for example, [36]), but also as expansion in space eigenfunctions (see [35]), etc.
In his numerous articles, Steklov investigated various eigenvalue problems essential for the Fourier method. Deep results about the asymptotic behavior of eigenvalues and eigenfunctions and about convergence of the corresponding series expansions were obtained in [29]. He also showed that the closedness of a system of eigenfunctions is of crucial importance. It is worth mentioning that Steklov presented the systematic rigorous justification of the Fourier method in the first volume of his monograph [33]. Initial-boundary value problems for parabolic and hyperbolic equations are considered in this monograph in the case when variable coefficients do not depend on the time but only on a single spatial variable. A detailed study of the Sturm–Liouville problem serves as the basis for this, and 6 of the 11 chapters of this volume are devoted to this problem.

The aim of the present article is to outline the development of the Fourier method in Russia before and after Steklov’s important contributions to this field. Indeed, it is practically unknown in present-day Russia that one of the first applications of the Gauss–Ostrogradsky formula (usually referred to as the divergence theorem in English textbooks) appeared in the note [22]. He also showed that the closedness of a system of eigenfunctions is of crucial importance. It is worth mentioning that Steklov presented the systematic rigorous justification of the Fourier method in the first volume of his monograph [33]. Initial-boundary value problems for parabolic and hyperbolic equations are considered in this monograph in the case when variable coefficients do not depend on the time but only on a single spatial variable. A detailed study of the Sturm–Liouville problem serves as the basis for this, and 6 of the 11 chapters of this volume are devoted to this problem.

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After Steklov’s death, much work on the Fourier method by A. N. Krylov, I. G. Petrovsky, O. A. Ladyzhenskaya, and many others resulted in substantial progress, in particular for the case of multiple spatial variables, which is an especially difficult case because smoothness of the boundary must be taken into account for a domain in m-dimensional Euclidean space. So far the deepest results for this case were obtained by O. A. Ladyzhenskaya [18] and V. A. I’In [7]. In the second section, we provide an idea of some results in this direction obtained in Leningrad in the 1930s through the 1950s. In the last section, we outline how to weaken the requirements on the data smoothness sufficient for justifying the Fourier method. The corresponding idea was proposed long ago by Aleksei Nikolaevich Krylov [3], but was developed in terms of modern mathematical tools only during the past three decades.

**Ostrogradsky’s Work on the Heat Equation**

Ostrogradsky contributed to various areas of mathematical physics—in particular, four of his papers are devoted to hydrodynamics, three articles deal with elasticity theory—but his major contribution concerns the propagation of heat in solids and fluids (see [22], [23], and [24]). Indeed, he was well prepared for this. During his studies in Paris between 1822 and 1827, he attended lectures by Laplace, Legendre, Binet, Cauchy, and, what is most important, by Fourier and Poisson. Before that, Ostrogradsky studied physics and mathematics at the University of Kharkov, but did not receive his degree there despite the fact that he passed all the necessary examinations. The refusal was “motivated” by his not attending lectures on philosophy and theology; this was the reason for his leaving Russia.

On his arrival to St. Petersburg in 1828 after studies in Paris, Ostrogradsky presented three notes to the Academy of Sciences (one of them was [22]). These scientific achievements and favourable reports of prominent French mathematicians resulted in his election (in December 1828) to the Academy as an adjunct. Less than 2 years later Ostrogradsky became an extraordinary academician; his election as full academician in applied mathematics took place in December 1831.

Let us turn to Ostrogradsky’s role in developing the theory of heat propagation. The best way to characterize it is to quote Steklov’s talk at a meeting in Poltava (Ostrogradsky’s birthplace in Ukraine) on the occasion of the centenary of Ostrogradsky’s birth (see [32]):

When the equation that describes the propagation of heat in a solid was established by Fourier, it became essential to develop methods for finding solutions that satisfy some prescribed boundary and initial conditions. Since this question is extremely difficult in its full generality, it was natural to begin with the simplest cases that demonstrate specific features of the problem.