Kolmogorov’s differential equations and positive semigroups on first moment sequence spaces

1. Introduction

Spatially implicit metapopulation models with discrete patch-size structure and host-macroparasite models which distinguish hosts by their parasite loads lead to infinite systems of ordinary differential equations. In several papers, a this-related theory will be developed in sufficient generality to cover these applications. In this paper the linear foundations are laid. They are of own interest as they apply to continuous-time population growth processes (Markov chains). Conditions are derived that the solutions of an infinite linear system of differential equations, known as Kolmogorov’s differential equations, induce a $C_0$-semigroup on an appropriate sequence space allowing for first moments. We derive estimates for the growth bound and the essential growth bound and study the asymptotic behavior. Our results will be illustrated for birth and death processes with immigration and catastrophes.
deaths, births and emigration of residents (release of parasites). They have the properties typical for infinite transition matrices in stochastic processes with continuous time and discrete state (continuous-time birth and death chains, e.g., see [1, 12, 13] and the references therein). Typically the sequence of diagonal coefficients \( \alpha_{kk} \) is unbounded. The function \( f \) gives the rate of change of the number of dispersers (free-living parasites) due to patch emigration, immigration and disperser death. The functions \( g_j \) describe the rate of change of the numbers of patches with \( j \) residents (hosts with \( j \) parasites) due to the immigration of dispersers (invasion of parasites). One possible approach to these equations, chosen by Karl–Peter Hadeler and collaborators [10, 16, 17, 28], derives and analyzes partial differential equations for the moment generating functions of \( x_j(t) \). This even works for infinite systems of partial differential equations and yields impressive and illuminating results, but requires the matrix \( \alpha_{jk} \) to be essentially tridiagonal and \( \alpha_{jk} \) to depend on \( j \) and \( k \) in a rather restricted way. It is our aim to develop a theory of semiflows on an appropriate sequence space which works without these restrictions [29, 30] and in particular to establish conditions for the solution semiflow to be dissipative [18], have a compact attractor for bounded sets [18, 39], and be uniformly persistent [19, 42, 4, 44]. We choose a somewhat more abstract approach than the ones in [2] and [4] from which we have received much inspiration in order to include a variety of models (in [29] we assume that only juveniles migrate) and to include state transitions which are not of nearest-neighbor type.

The biological interpretation gives us guidance how to choose the appropriate state space [2]. Assuming that meaningful solutions are non-negative, the number of patches (hosts) is given by \( \sum_{j=0}^{\infty} x_j \) and the number of residents (in-host parasites) by \( \sum_{j=1}^{\infty} j x_j \). In a population growth process, \( \sum_{j=0}^{\infty} x_j = 1 \) and \( \sum_{j=1}^{\infty} j x_j \) is the expected population size. We recall the standard sequence space notation

\[
\ell^1 = \left\{ (x_j)_{j=0}^{\infty}; \ x_j \in \mathbb{R}, \sum_{j=0}^{\infty} |x_j| < \infty \right\}
\]

(2)

with norm

\[
\|x\| = \sum_{j=0}^{\infty} |x_j|, \quad x = (x_j)_{j=0}^{\infty}.
\]

(3)

and introduce the first-moment space [2]

\[
\ell^{11} = \left\{ (x_j)_{j=0}^{\infty}; \ x_j \in \mathbb{R}, \sum_{j=0}^{\infty} j |x_j| < \infty \right\}
\]

(4)

As norm on \( \ell^{11} \) we choose

\[
\|x\|_1 = \sum_{j=0}^{\infty} (1 + j) |x_j|, \quad x = (x_j)_{j=0}^{\infty}.
\]

(5)