Generalized eikonalization and unitarity

M. Giffon\textsuperscript{1}, E. Martynov\textsuperscript{2}, E. Predazzi\textsuperscript{3}

\textsuperscript{1} Institut de Physique Nucléaire de Lyon Université Claude Bernard, F-69622 Villeurbanne Cedex, France (e-mail Giffon@frcpn11.in2p3.fr)
\textsuperscript{2} Bogoliubov Institute for Theoretical Physics National Academy of Science of Ukraine – 252143 Kiev – Ukraine (e-mail martynov@gluk.apc.org)
\textsuperscript{3} Dipartimento di Fisica Teorica dell' Università di Torino – Italy and Sezione INFN di Torino – Italy (e-mail predazzi@to.infn.it)

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Abstract. In the literature, the notion of eikonalization is often used as synonymous of unitarization or, at least, as implying that unitarity is not violated. This, to the very least, appears to be wishful thinking. We discuss the properties of various types of eikonalization within a unified treatment. Linear trajectories with intercept larger than unity (so popular nowadays) lead to small asymptotic violations of unitarity even after eikonalization. Classes of eikonalizations in which the Odderon could dominate over the Pomeron are given; even so the maximal Odderon is still excluded by eikonalization.

1 Introduction

Ever since it was introduced as a useful approach to high energy physics, eikonalization was recognized as a useful tool to alleviate the violations of unitarity which could possibly be induced by some Ansatz tailored according to the prescriptions of some physical model. The boundaries between this approach and a true unitarization have, however, never really been clarified entirely; a common statement is that eikonalization is a way to take into account some properties of high energy s-channel unitarity.

While we do not believe that one could answer in full generality how eikonalization and unitarity are related, we will clarify quite a few points. Working on specific examples we will show that not only eikonalization does not mean unitarization (this is trivial), but that in some important cases, even after eikonalizing, unitarity can still be (mildly) violated.

We will first review several of the eikonalization procedures that have been proposed in the past. This will allow us to classify the various methods, to study them in a unified way and, also, to derive some new interesting relations between different (and apparently disconnected) ways of eikonalization. In particular, we will show that a Borel transform relationship exists between what we will call Quasi Eikonalization (QE) and U-Matrix Eikonalization (UME). This, we will do by extending our considerations to larger classes of eikonalization which we will call Generalized Eikonalization (GE).

In addition to the previously quoted ones, our main results are the following:

i) Any model for input Pomeron (or Odderon), be it simple pole, multipole or cut with intercept larger than unity but with linear trajectory for its leading contribution, violates asymptotically unitarity even after eikonalization and cannot satisfy the asymptotic constraints required on the impact parameter amplitude.

ii) We will point to two extreme classes of eikonalization: one (similar to the QE) where the Pomeron always must dominate over the Odderon for unitarity not to be violated and one (similar to the UME) where the Odderon may dominate over the Pomeron in a large interval of the impact parameter variable.

iii) A Maximal Odderon is always excluded as it leads to asymptotic violation of unitarity (as already shown previously for more restricted classes of eikonalization).

2 Methods of eikonalization

Several approaches to the impact parameter representation of high energy amplitudes have been suggested in the past [1]. As it turns out, it is indeed very convenient and useful to investigate the properties of the scattering amplitudes at high energy in the impact parameter representation (among other things, it is in this picture that one best displays possible saturations or violations of unitarity, see, for instance, [2]).

In order to fix the notation, let us introduce the scattering amplitudes for the elastic $pp$ and $\bar{p}p$ processes, $M_{pp}^{\bar{p}p}(s,t)$. The corresponding impact parameter amplitudes are

$$H_{pp}^{\bar{p}p}(s,b) = \int \frac{d^2q}{2\pi} e^{iqb} M_{pp}^{\bar{p}p}(s,t),$$

(2.1)

where $s$ and $t$ denote the usual Mandelstam variables and we assume the process to occur in the physical s-channel; we also have $t = -q^2$. These amplitudes must satisfy the unitarity condition

$$\Re m H(s,b) = |H(s,b)|^2 + G_{\text{in}}(s,b),$$

(2.2)
where $G_{nn}(s, b)$ describes the contribution of inelastic processes where most of our ignorance resides. In the following, we will just use $G_{nn}(s, b) \geq 0$. Our normalization will be chosen so that

$$\sigma_{\text{tot}} = 8\pi \Im M(s, 0),$$

and

$$\frac{d\sigma}{dt} = 4\pi |M(s, t)|^2.$$

Next, we introduce the crossing-even and crossing-odd amplitudes defined as

$$M^\pm (s, t) = \frac{1}{2} [M^{pp}(s, t) \pm M^{pp}(s, t)].$$

In phenomenological approaches to high energy hadronic reactions, the procedure (alternative to the use of the complex angular momentum language) is the following. Physical principles and theoretical restrictions are first used to prepare an input amplitude which, for brevity, we will call the Born approximation (BA) to the physical amplitude. Normally, this Born approximation satisfies general requirements and responds to some physical expectations but, in general, will ultimately lead to some kind of violation of unitarity. The common remedy invoked at this point, is to eikonalize the BA. Beginning from the simple recipe mentioned earlier [1], various generalizations have been offered; in the following we will briefly analyze the Quasi Eikonalization (QE) [3,4] and the $U$-Matrix Eikonalization (UME) [5]. Details of these procedures can be found in the original references (phenomenological applications are too numerous for us to be able to give a comprehensive literature). In both approaches, the BA in the impact parameter variables, will be denoted by $h_+(s, b)$ (for the crossing-even part) and $h_-(s, b)$ (for the crossing-odd part). Asymptotically, they are just the Pomeron and the Odderon contributions respectively. Sub-asymptotically, other contributions may be very important for practical purposes but we will not be concerned with these aspects of the problem here.

The QE procedure leads to the following form of output amplitudes in terms of the input ones $h_{\pm}(s, b)$

$$H^{\mp}_{pp}(s, b) = \frac{1}{2ic} \exp(2iC\tilde{h}^{\mp}_{pp}(s, b) - 1),$$

where

$$\tilde{h}^{\mp}_{pp}(s, b) = h_+(s, b) \pm h_-(s, b).$$

For $C = 1$ we recover the usual eikonal form of the amplitudes.

The UME leads to a different relation between the input and the output amplitudes

$$H^{\mp}_{pp}(s, b) = \frac{h^{\mp}_{pp}(s, b)}{1 - 2iC h^{\mp}_{pp}(s, b)}.$$

In the original UME approach [5], the value of the parameter $C$ was taken $C = 1/2$. In what follows, for more generality, we will consider $C$ arbitrary (its value will just be constrained by unitarity to be $C \geq 1/2$ [6] for both (2.3) and 2.5).

As it turns out, both procedures, in their fullest generality can be treated from a unifying point of view, by considering each of them as a particular case of an even more general representation of $H(s, b)$. For this, we write the output amplitude $H$ in terms of the input one $h$ in what we will call the Generalized Eikonal (GE) which we define as

$$H^{\mp}_{pp}(s, b) = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{G(n)}{n!} [2i h^{\mp}_{pp}(s, b)]^n,$$  \hspace{1cm} (2.6)

with $G(1) = 1$.

Equation (2.6) is the sum of all multiple exchanges of Pomerons and Odderons with some weight function $G(n)$ for an $n$-reggeon contribution. $G(n)$ takes into account the deviation of the “true” $n$-reggeon contribution from the “optical” approximation to it where the hadrons in the intermediate states are on the mass shell [3, 4].

With $G(n) = C^{n-1}$ we obtain the QE expression (2.3); the case $G(n) = n! C^{n-1}$ gives the UME result (2.5) for the output amplitude. In its greatest generality, the function $G(n)$ may depend also on the energy $\sqrt{s}$ and on the impact parameter $b$. For simplicity, however, we will limit our consideration to the simple case when $G(n)$ depends just on $n$.

The series in (2.6) may have a finite radius of convergence which we will denote by $R$

$$R = \lim_{n \to \infty} (n+1)! G(n+1)/n! G(n+1) = \lim_{n \to \infty} (n+1) G(n)/G(n+1).$$

(2.7)

In the UME approach (2.5) the series (2.6) converges for $|h(s, b)| < 2e$. With a supercritical Pomeron (and, possibly, a supercritical Odderon), the input amplitude $|h(s, b)| \to \infty$ when $s \to \infty$. In this case, it will be necessary to provide an analytic continuation of the series (2.6). This can be done by using Borel’s summation. If, for a given $G(n)$, the limit (2.7) is finite, we define the function

$$\tilde{H}(z) = \int_0^\infty dt e^{-zt} \Phi(tz),$$

where

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{G(n)}{n!} x^n.$$

The function $\Phi(x)$ is called the Borel function associated to the series (2.6) for $H(z)$. The function $\tilde{H}(z)$ coincides with $H(z)$ inside the domain of convergence of the series (2.6) but will, in general, be defined in a larger domain of the variable $z$.

As a byproduct of this analysis, we notice that the two general eikonalization procedures considered so far, the QE and the UME, are related to one another. The QE series (see (2.3) or (2.6) with $G(n) = C^{n-1}$) is just the Borel function associated with the UME series of (2.5) (or of (2.6) with $G(n) = n! C^{n-1}$).

At this point, it is fit to remark once more that none of the procedures of eikonalization so far introduced leads to true unitarization. An eikonalization procedure will just remove the rougher violations of unitarity (for example restoring the Froissart-Martin bound for $\sigma_{\text{tot}}$ [7]), but this will not mean that the resulting amplitudes satisfy unitarity. We will