Bifurcations to Periodic Solutions in a Production/Inventory Model

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Summary. Total production costs sometimes show an S-shaped form. There are several ways in which a plant with given capacity can be adapted to a specific demand rate, one of them being adaptation of intensity per work hour. In this paper we present an application of the Hamilton-Hopf bifurcation to an inventory/production intensity splitting model with a nonconvex cost function. Our analysis provides a new proof that persistent oscillations may be optimal for arbitrary small discount rates. For zero discounting a “Hamilton Hopf bifurcation” occurs, leading to a family of periodic solutions bifurcating from a steady state. If the discount rate becomes positive, almost all periodic solutions vanish; only a unique branch of periodic solutions is obtained.

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1. Introduction

Usually, machines or production plants are constructed for a certain production intensity. In a neighbourhood of that intensity there is an efficient way of production, whereas any deviation below or above those levels leads to decreasing efficiency of production, i.e., to increasing unit production cost. Under those conditions the unit production costs show a U-shaped form. Writing these costs as a function of the production intensity \( v \), i.e.,
ϕ(v), the (total) production cost, Φ(ν), may be written as

\[ \Phi(ν) = ν ϕ(ν). \]

These costs typically show a concave-convex form.

We consider a production facility operating at production speed ν. The production cost \(\Phi(ν)\) behaves as indicated in Figure 1: There are two ranges of efficient production speeds, one at low level (“do nothing”) and one at normal level (“operate regularly”). In between these levels, \(\Phi(ν)\) is nonconvex, indicating suboptimal resource utilization.

Let us assume that the produced good sells at a constant demand rate \(d\) and the production beyond \(d\) is stored in inventory. If the production rate \(v\) is below \(d\), the demand is satisfied from the storage. If the inventory is exhausted and there is still an excess demand, it is backlogged, i.e., shortages are allowed. Since only small oscillations about the equilibrium state \(v_0 = d\) are investigated, we need not explicitly restrict the production speed to nonnegative values. These restrictions would have to be imposed if the numerically calculated large-amplitude motion shows negative production speeds.

When the production intensity can be adapted instantaneously and without cost, a policy of switching intensity between a high and a low level turns out to be optimal (“intensity splitting”). If the adjustment of production intensity is charged with costs and inventories are allowed, then even a constant demand can lead to persistent oscillations, i.e., to gradual intensity-splitting of production rates.

A well-known result of production planning says that nonconvexities of the production cost function may lead to intensity splitting. This production pattern means that the speed of the production, i.e., its intensity, varies within a certain interval. Intensity splitting due to concave production costs occurs both in statistic and dynamic production planning. For a detailed discussion of the latter case, compare Feichtinger and Sorger [14] and Feichtinger et al. [13].