An extension of McGarvey’s theorem from the perspective of the plurality collective choice mechanism

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Abstract In 1953, David McGarvey showed that if the number of voters is unrestricted, then the set of outputs obtained from majority rule is a very general class of binary relations. We will present an extension of McGarvey’s Theorem based on the plurality collective choice mechanism.

1 Introduction

In 1953, David McGarvey showed that if the number of voters is unrestricted, then it is possible to obtain an arbitrary asymmetric binary relation as the output of majority rule (McGarvey 1953). McGarvey’s seminal result has been previously extended in several different directions, including the cases of separable preferences (Hollard and Le Breton 1996) and λ-majorities (Mala 1999). Our study generalizes McGarvey’s theorem in a different direction with an emphasis on social choice.

The idea is to identify majority rule with the plurality choice function restricted to the family of all 2-element subsets of the set of alternatives. Recall that for a given profile of linear orders the plurality choice function outputs a set of winners where each winner receives the maximum number of first place votes. In other words, an alternative \( x \) is a winner if the number of first place votes for any other feasible alternative is less than or equal to the number of first place votes for \( x \). If the plurality choice function is restricted to a 2-element subset \( \{x, y\} \), then the output is either \( \{x\} \) (more voters prefer \( x \) over \( y \)), \( \{y\} \) (more prefer \( y \)), or \( \{x, y\} \) (alternatives \( x \) and \( y \) are equally preferable). Now suppose we specify the output first. In other words, for each 2-element subset of alternatives select arbitrarily a 1-element subset or the 2-element set itself. If \( n \) is the number of alternatives, then the selection process described above...
gives an \( \binom{n}{2} \) collection \( B \) of subsets of alternatives. By McGarvey’s theorem, there exists a profile \( P \) of linear orders such that the output of the plurality choice function restricted to all 2-element subsets of the set of alternatives is the given collection \( B \).

We extend McGarvey’s theorem to the case where the plurality choice function is restricted to the set of all \( \ell \)-element subsets of alternatives where \( \ell \) can be any integer greater than 1 and less than \( n \). This result is actually a corollary of one of Saari’s results on voting dictionaries (Saari 1989). Saari deals with rankings instead of choice sets and he considers a whole class of tally procedures of which plurality voting is just one. So the contribution given here is not so much in the statement of our result (even thought it is interesting in its own right), but in the fact that our proof of it is constructive and it reflects McGarvey’s style of proof. Moreover, we go on to show that (even thought it is interesting in its own right), but in the fact that our proof of it is just one. So the contribution given here is not so much in the statement of our result sets and he considers a whole class of tally procedures of which plurality voting is restricted to the set of all \( \ell \)-element subsets of alternatives where \( \ell \) represents the set of all \( \ell \)-element subsets of alternatives where \( \ell \) can be any integer such that any possible outcome of the plurality choice function restricted to the family of \( \ell \)-element subsets of alternatives may be realized by a profile of length at most \( k \).

2 Collective choice mechanisms and an extension of McGarvey’s theorem

In this section, in addition to introducing notation and terminology, we will state and prove the first of our two results.

Let \( X \) be a finite set of alternatives and assume that \(|X| \geq 3\). A binary relation \( \rho \) on \( X \) is asymmetric if \((x, y) \in \rho\) implies that \((y, x) \notin \rho\) for all \( x, y \in X \). The binary relation \( \rho \) is complete if \((x, y) \notin \rho\) implies that \((y, x) \in \rho\) for all \( x \neq y \) in \( X \). Next, \( \rho \) is transitive if \((x, y) \in \rho\) and \((y, z) \in \rho\) implies that \((x, z) \in \rho\) for all \( x, y, z \in X \). An asymmetric, complete, and transitive binary relation on \( X \) is called a linear order on \( X \). We will represent a linear order on \( X \) as \( xyz \ldots \) where \( x \) is ranked first, \( y \) is ranked second, and so on. If \( \mathcal{L}(X) \) is the set of all linear orders on \( X \) and \( \mathcal{A}(X) \) is the set of all asymmetric binary relations on \( X \), then \( \mathcal{L}(X) \subseteq \mathcal{A}(X) \). For any integer \( k \geq 1 \), a \( k \)-tuple \( P = (P_1, \ldots, P_k) \) of linear orders on \( X \) is called a profile and \( k \) is the profile length of \( P \). The collection \( \bigcup_{k \geq 1} \mathcal{L}(X)^k \) represents the set of all profiles with no restriction on profile length. For any two profiles \( P = (P_1, \ldots, P_k) \) and \( P' = (P'_1, \ldots, P'_j) \) we define the sum of \( P \) and \( P' \) as follows:

\[
P + P' = (P_1, \ldots, P_k, P'_1, \ldots, P'_j).
\]

So the profile \( P + P' \) has length \( k + j \).

Let \( A \) be a nonempty family of nonempty subsets of \( X \). Then

\[
\emptyset \neq A \subseteq 2^X \setminus \{\emptyset\}.
\]

A collective choice mechanism over \( A \) is a countable collection \( C = (C^k)_{k \geq 1} \) where \( C^k \) is a mapping from \( A \times \mathcal{L}(X)^k \) into \( 2^X \) such that for all \( A \in A \) and for any profile

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