Sophisticated voting and equilibrium refinements under plurality rule

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Abstract. In this paper we show in the context of voting games with plurality rule that the “perfect” equilibrium concept does not appear restrictive enough, since, independently of preferences, it can exclude at most the election of only one candidate. Furthermore, some examples show that there are “perfect” equilibria that are not “proper”. However, also some “proper” outcome is eliminated by sophisticated voting, while Mertens’ stable set fully satisfies such criterium, for generic plurality games. Moreover, we highlight a weakness of the simple sophisticated voting principle. Finally, we find that, for some games, sophisticated voting (and strategic stability) does not elect the Condorcet winner, neither it respects Duverger’s law, even with a large number of voters.

1 Introduction

An interesting application of strategic stability is offered by voting games. As well known, the solution concept of Nash equilibrium has been proved to be too weak to fully capture strategic behavior of the players. Different solution concepts have been proposed to eliminate this weakness. However, all such solution concepts have been proven to be equivalent for generic normal form games. The peculiar aspect of a voting game is given by the fact that its normal form representation is highly non generic, as the same outcome arises

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from many different pure strategy combinations. More precisely, given the set of players \( N = \{1, \ldots, n\} \), the set of candidates \( K = \{1, \ldots, k\} \) and the voting procedure \((VP)\), we have a game form. To obtain the associated non-cooperative game we need the collection of utility vectors \( \{u^i\}_{i \in N} \), where \( u^i = (u^i_1, \ldots, u^i_k) \) and each \( u^i_c \) represents the payoff that player \( i \) gets if candidate \( c \) is elected. In other words, the triplet \((N, K, VP)\) defines a family of games, each game in this family is identified with \( nk \) numbers; e.g. with plurality rule each voter has \((k + 1)\) pure strategies, therefore the corresponding normal form has \( n(k + 1)^n \) numbers. Hence results proven in the general class of normal form games have to be reconsidered in this context, since the voting procedure imposes constraints on the corresponding class of normal form games.

In this paper we study the behavior of three solution concepts (“perfect” equilibrium, “proper” equilibrium and Mertens’ stable set) in the class of games defined by plurality rule and we relate them to the sophisticated voting principle. To do so, we apply the general model of a one stage voting procedure defined by Myerson and Weber [19] to the plurality rule case. Given the set of candidates \( K = \{1, \ldots, k\} \), each voter submits a ballot which is a vector of \( k \) components. An electoral system is then defined by the set of possible ballots that each voter can submit and by the election rule that, given the cast ballots, selects the winner from the set \( K \). Hence, with the plurality rule every voter has the same strategy space and each pure strategy is a vector with all zeros except for a one in position \( c \) which represents the vote for candidate \( c \). Abstention can be represented by the zero vector. With plurality, the election rule selects the candidate that receives the largest total number of votes. In case of ties, we allow an equal probability lottery among the winners.

The importance of refining the Nash concept in voting games with plurality follows from the trivial observation that, in such games, even if every voter has the same preference order on the various alternatives, voting for the least preferred candidate is a Nash equilibrium if there are more than three voters. This follows from the main drawback of this concept, i.e. that it admits the use of (weakly) dominated strategies. We show that the solution concept of “perfect” equilibria, even if it guarantees admissibility, is not restrictive enough in this context, since it can exclude at most the election of only one candidate (a Condorcet loser), independently of the voters’ preferences. Simple examples show that the “proper” equilibrium is a refinement of the “perfect” concept, even for plurality games, but also that there are cases where some outcome selected by properness is eliminated by sophisticated voting and is not induced by any stable set. At this point it is important to stress that the different behavior of the various refinements considered holds in complete neighborhoods, in the space of plurality games, of the proposed examples and hence these cannot be considered pathological.

We show in [5] that, for generic utility vectors, the set of equilibria which induce a mixed distribution over the outcomes (i.e. with two or more candidates elected with positive probability) is finite and furthermore each of these is regular (in Harsanyi’s sense). This result implies that, for generic plurality games, there are only finitely many equilibrium distributions and furthermore,