The $C^1$ topology on the space of smooth preference profiles*

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Abstract. This paper defines a fine $C^1$-topology for smooth preferences on a “policy space”, $W$, and shows that the set of convex preference profiles contains open sets in this topology.

It follows that if the dimension $W \leq v(\mathcal{D}) - 2$ (where $v(\mathcal{D})$ is the Nakamura number of the voting rule, $\mathcal{D}$), then the core of $\mathcal{D}$ cannot be generically empty. For higher dimensions, an “extension” of the voting core, called the heart of $\mathcal{D}$, is proposed. The heart is a generalization of the “uncovered set”. It is shown to be non-empty and closed in general. On the $C^1$-space of convex preference profiles, the heart is Paretoian. Moreover, the heart correspondence is lower semi-continuous and admits a continuous selection. Thus the heart converges to the core when the latter exists. Using this, an aggregator, compatible with $\mathcal{D}$, can be defined and shown to be continuous on the $C^1$-space of smooth convex preference profiles.

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1. Introduction

The core of a voting game (without vetoers) on a compact policy space $W$ is non-empty only for a nowhere dense set of preferences, when preferences are continuous and endowed with the closed convergence or $C^0$-topology: that is emptiness of the voting core is generic in this topology (see Le Breton [17]). On the other hand if preferences are continuous and convex and the dimension of the space is suitably bounded then a core does exist ([30]).

These two results are compatible since the set of convex preferences is nowhere dense in the $C^0$-topology on preferences. This paper considers the set of smooth “Morse” (non-degenerate) preference profiles and constructs a $C^1$-topology for this set (Sect. 3). Essentially the topology uses information on the gradients defined for representative utilities. In this topological space, the subset of convex preferences contains open sets of preferences. It is therefore possible to show, as long as the dimension of the policy space is bounded in an appropriate fashion, that emptiness of a voting core cannot be generic (in the $C^1$-topology) on utility profiles (Theorem 1) or preference profiles (Theorem 2). However if the dimension of $W$ is sufficiently high, then emptiness of a voting core is generic in the $C^1$-topology (Theorem 3). This result holds true in the $C^1$-subspaces of Euclidean and convex preference profiles (Corollary 1).

Since the core of a majority voting rule is generically empty even in three dimensions, an “equilibrium” notion called the uncovered set has been proposed (see [22] and [8]). This set contains the support of mixed strategy Nash equilibria in two-party competition with vote-maximizing candidates. Moreover this set “approximates” the core when the latter exists. However the uncovered set uses global information on preferences and is generally difficult to compute.

Section 4 defines a local version of the uncovered set, termed the “heart”. It is shown (Theorem 4) that the heart of a voting rule for a smooth preference profile is always non-empty on a compact space, $W$. Moreover, the heart (regarded as a correspondence from the space of smooth preference profiles to $W$) is lower hemi-continuous on the subspace of convex profiles, endowed with the $C^1$-topology. In addition, the heart, at a given profile, belongs to the Pareto set of the profile (Corollary 2). This result implies that if the voting core is non-empty at a profile, $p$, and preferences converge to $p$, then the heart converges to the core. Thus the heart is a continuous “extension” of the core. The existence and Pareto property of the heart replicate the same properties of the uncovered set.

The lower hemi-continuity and selection property of the heart correspondence allow it to be used to define an aggregator (mapping to social preference) which is compatible with the voting rule, and is continuous (in the $C^1$-topology) on convex preference profiles. Section 4 concludes with a remark on the possibility of constructing an aggregator which is “generically” continuous on the full space of smooth Morse preference profiles. Section 5 briefly discusses the application of the notion of the heart in modeling political compe-