Condorcet efficiencies under the maximal culture condition

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Abstract. The Condorcet winner in an election is a candidate that could defeat each other candidate in a series of pairwise majority rule elections. The Condorcet efficiency of a voting rule is the conditional probability that the voting rule will elect the Condorcet winner, given that such a winner exists. The study considers the Condorcet efficiency of basic voting rules under various assumptions about how voter preference rankings are obtained. Particular attention is given to situations in which the maximal culture condition is used as a basis for obtaining voter preferences.

1. Introduction

Consider an election on three alternatives (A, B, and C). There are six possible rankings that each of n (odd) voters might have on these candidates.

\[
A \quad A \quad B \quad C \quad B \quad C
\]
\[
B \quad C \quad A \quad A \quad C \quad B
\]
\[
B \quad C \quad A \quad B \quad A \quad A
\]
\[
n_1 \quad n_2 \quad n_3 \quad n_4 \quad n_5 \quad n_6
\]

Here, \(n_i\) denotes the number of voters having the associated preference ranking, with \(n = \sum_{i=1}^{6} n_i\). Since each of the six rankings represents a linear preference order, voter indifference between candidates is not allowed. A specific combination of \(n_i\)'s is referred to as a voter profile, or simply as a profile.

A candidate is the Condorcet winner if it would be able to defeat each of the other two candidates in a series of pairwise majority votes. Thus, candidate A would be the Condorcet winner if \(n_1 + n_2 + n_3 > n_4 + n_5 + n_6\) (A beats C by majority rule) and if \(n_1 + n_2 + n_4 > n_3 + n_5 + n_6\) (A beats B by majority...
rule). It is well known that a Condorcet winner does not necessarily exist (Condorcet 1785). However, when such a candidate does exist, the Condorcet criterion suggests that it should be elected as the winner of the election.

The Condorcet efficiency of a voting rule is the conditional probability that the voting rule will elect the Condorcet winner, given that a Condorcet winner exists. The purpose of the current study is to examine the Condorcet efficiency of some common voting rules, when voter profiles meet the maximal culture condition, which will be defined later. Particular attention will be given to general observations that can be made regarding the Condorcet efficiency of common voting rules over three different previously established conditions for obtaining voter profiles.

2. Profile generating methods

Since the notion of Condorcet efficiency involves conditional probabilities of events related to voter profiles, some assumption must be made about the likelihood that various profiles will occur. If some candidate has a relatively high likelihood of being preferred by most voters, then that candidate will be likely to be elected by any voting procedure that is used. Thus, the most interesting cases are situations in which there is a balance, or neutrality, in voters’ preferences, to reflect situations in which the selection of the voting rule to be used is likely to be most critical in determining the outcome of the election. Gehrlein and Fishburn (1981) define three different methods of describing how voter profiles might be randomly generated. Each of these profile generating methods is neutral toward candidates and tends to have an overall expected balance of voter preferences, to suggest the case of close elections. The three profile generating methods are:

*Impartial Culture (IC)* – Each of the $n$ voters is independently assigned a preference ranking. Each of the six possible preference rankings is equally likely to be assigned to the voters.

*Impartial Anonymous Culture (IAC)* – Each voter profile with a combination of $n_i$’s that sum to $n$ is assumed to be equally likely to be observed.

*Maximal Culture (MC)* – A positive integer, $L$, is selected, and each $n_i$ is drawn from a uniformly random distribution on the integers $\{0, 1, 2, 3, \ldots, L\}$. Unlike IC and IAC, MC does not have a fixed number of voters.

We are primarily interested in considering situations for which closed form representations can be developed for the probabilities in question. As a result, we shall generally only consider the limiting case in voters ($n \to \infty$) for IC.

IC and IAC are special forms of Pólya-Eggenberger (P-E) urn models (Berg 1988). P-E models describe a family of discrete multivariate contagion probability models. To describe them in the context of this particular example, consider an urn containing six balls of different colors. Each of the six